Vector Measures and Stochastic Integration

By

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To Dot

Whose patience and encouragement made this possible.

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TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate Council of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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This dissertation investigates the stochastic integration of scalar-valued functions from the point of view of vector measure and integration theory. We make a detailed study of abstract integration theory in Chapters I and II, and then apply our results in Chapter III to show that certain kinds of stochastic integrals, previously defined by other means, are special cases of the general theory. In carrying out this program we prove extended forms of the classical convergence theorems for integrals. We also establish a generalization of the standard extension theorem for scalar measures generated by a left continuous function of bounded variation. The special case of measures in Hilbert space is discussed, and a corrected form of a theorem of Cramér is proved. In Chapter III we show that certain sample path integrals, the Wiener-Doob integral, and a general martingale integral are included in the abstract integration theory. We establish a general existence theorem for stochastic integrals with respect to a martingale in L_p, $1 < p < \infty$.

INTRODUCTION

This dissertation concerns stochastic integration, a topic in the theory of stochastic processes. A stochastic process (random function) is a function from a linear interval T into an L_p space over a probability measure space. In its broadest terms, stochastic integration deals with linear transformations from a class of functions x to a class of stochastic process $\int x dz$, which depend on a fixed process z. The functions x which are transformed may be sure (scalar-valued) functions, or more generally, random functions. Since random functions take their values in an L_p space they are vector-valued functions. Hence we are in the framework of the general theory of bilinear integration developed by Bartle [1]. The vector-valued function z generates a finitely additive measure on an appropriate ring or algebra of sets. When the integrand x is a sure function we are concerned with the integration of scalar functions by vector measures. When the integrand is another random function we have the full generality of the Bartle Theory, where both the integrand and the measure take values in Banach spaces X and Z respectively, and there is a continuous bilinear multiplication from $X \times Z$ into another Banach space Y, in which the integral takes its values.

Until recently the field of stochastic integration appeared to be in much the same state as the area of real function theory was before the systematic introduction of measure theory. Several relatively unrelated kinds of stochastic integrals can be found throughout the literature on stochastic processes and their applications. Most of these integrals are defined as limits of approximating sums of the Riemann-Stieltjes-type. This approach, however, is not well-suited to stochastic integration in general, and certain difficulties may arise. With the exception of some work by Cramér [10], the measure properties inherent in the stochastic integration scheme have not been exploited.

There is currently a great deal of interest in unifying the field of stochastic integration using vector measure and integration theory. E. J. McShane, who has been investigating stochastic integration for several years, is presently writing a book which should contribute to this unification. The recent Symposium on Vector and Operator-valued Measures and Applications (Salt Lake City, August, 1972) was specifically concerned with stochastic integration, and attracted many of the known experts in both the fields of measure and integration theory (Brooks, Dinculeanu, Ionescu Tulcea, Kelley, and Robertson) and stochastic processes and stochastic integration (Chatterji, Ito, Masani, and McShane). A very recent unpublished paper by Metivier [23], presented at the Symposium, deals with the problem of incorporating a rather general kind of stochastic integral within the framework of the Bartle integration theory.

The purpose of this dissertation is more modest. We confine our attention to the case of scalar-valued integrands, and present a detailed investigation and synthesis of the appropriate vector measure and integration techniques, which are then used to exhibit certain kinds of stochastic integrals as special cases. In carrying out this program we make improvements in known results concerning the convergence of sequences of integrals, measures generated by vector functions, and existence of stochastic integrals.

Chapter I develops the abstract integration theory of Bartle [1] for the special case at hand, namely, the integration of scalar functions by finitely additive vector measures. In this setting more is true than in the general Bartle theory. For example, Theorem I.5.8 states roughly that the integral commutes with any bounded linear operator. This fact enables us to prove extended forms of the classical convergence theorems, and sketch a theory of L_p spaces. We discuss briefly the case of integration by countably additive vector measures, and introduce another integral developed by Gould [16]. This integral is defined by a net of Riemann-Stieltjes-type sums, as opposed to the Lebesgue approach used in the development of the Bartle integral by means of simple functions. We show that these two integrals are equivalent - a fact not discussed by Gould.

Since a stochastic process is a function from an interval T of the line into an L_p space, as will be discussed in Chapter III, the stochastic integral of a scalar function f on T

with respect to z can be thought of as a Stieltjes-type integral $\int_T f(t)z(dt)$ in L_p. With this in mind, we investigate in Chapter II the properties of a Stieltjes measure m generated by a vector function z on T. We discuss the boundedness of m, and the question of the existence of a countably additive extension to the Borel sets of T. For a rather general class of Banach spaces we show in Theorem II.3.9 that the classical extension theorem for scalar measures on an algebra is valid. We also give a counter-example to show that the theorem fails to hold in the Banach space c_o. Finally we discuss the special case when z takes its values in a Hilbert space. We consider a theorem of Cramér [10], which is incorrect as stated, and apply our results to establish a corrected version. Some implications of this theorem are then discussed.

In Chapter III we first present some background information on probability and stochastic processes, and then motivate the study of stochastic integration. We show that the general integration theory of Chapters I and II includes certain sample path stochastic integrals, the Wiener-Doob integral, and a martingale stochastic integral. Several related results are also discussed.

CHAPTER I

Abstract Integration

In this chapter we discuss the integration of scalar functions by finitely additive vector measures. Section 1 establishes the notation we will use, while Sections 2 and 3 list some basic properties of vector measures. In Section 4 we discuss the space of measurable functions and convergence in measure. Section 5 is concerned with integration theory proper, and in Section 6 we apply this theory to prove convergence theorems for integrals. Section 7 outlines part of the theory of L_p spaces in this setting. Countably additive measures are discussed briefly in Section 8. Finally, in Section 9, we compare a Riemann-type integral with the Lebesgue-type integral of Section 5, and establish the equivalence of these two approaches.

1. Notation. Throughout this dissertation the following terminology and notation will be used. N is the set of natural numbers $\{1,2,3,\ldots\}$ and $N_n = \{1,2,\ldots,n\}$. The lower-case letters i,j,k,ℓ, and n will always denote elements of N. Δ is a finite subset of N. R denotes the real number field; we use the usual notation for intervals in R, so that (a,b) is an open interval, $(a,b]$ a half-open interval, and so on.

X and Y are Banach spaces over the scalar field Φ, which may be either the real or the complex field unless otherwise specified. $|x|$ denotes the norm of an element $x \in X$. X_1 is the unit ball of X, that is, $X_1 = \{x \in X: |x| \leq 1\}$. $B(X,Y)$ is the Banach space of bounded linear transformations from X to Y, with the usual topology of uniform convergence. $X^* = B(X,\Phi)$, and X_1^* is the unit ball of X^*. Recall that by definition,

$$|x^*| = \sup_{X_1} |x^*x| ,$$

while by the Hahn-Banach Theorem,

$$|x| = \sup_{X_1^*} |x^*x| .$$

Suppose that $\mathcal{R}$ and Σ are families of subsets of a non-empty set S. $\mathcal{R}$ is a ring if $\mathcal{R}$ is closed under the formation of relative complements and finite unions. A ring $\mathcal{R}$ is a σ-ring if $\mathcal{R}$ is closed under countable unions. Σ is an algebra if Σ is a ring and $S \in \Sigma$. An algebra Σ is a σ-algebra if Σ is closed under countable unions. If $\mathcal{S}$ is any family of subsets of S, then $\sigma(\mathcal{S})$ denotes the smallest σ-algebra containing $\mathcal{S}$. If $E \in \mathcal{R}$, then $\pi(E,\mathcal{R})$ denotes the family of all partitions $P = \{E_i : 1 \leq i \leq n\}$ of E whose sets belong to $\mathcal{R}$. When $\mathcal{R}$ is understood we write $\pi(E)$. If (A_i) is a sequence of subsets of S which are pairwise disjoint, we call (A_i) a disjoint sequence. Whenever we write $A = \Sigma A_i$ instead of $A = \cup A_i$ for some set A and sequence (A_i), we mean that (A_i) is a disjoint sequence, and say that A is the sum of the A_i's. Similarly we write $A + B$ instead of $A \cup B$, if $A \cap B = \emptyset$.

Suppose that $\mathcal{R}$ is a ring and $m:\mathcal{R} \to X$ is a set function. m is finitely additive if $m(A + B) = m(A) + m(B)$. m is countably additive if $m(\Sigma A_i) = \Sigma m(A_i)$ whenever (A_i) is a sequence in $\mathcal{R}$ such that $\Sigma A_i \in \mathcal{R}$. We make the convention that whenever we write $m(A)$ it is assumed that A belongs to the domain of m. A finitely additive set function defined on a ring or algebra will be called a measure.

If $m:\mathcal{R} \to [o,\infty]$ is a set function, then m is monotone if $m(A) \leq m(B)$ whenever $A \subseteq B$ and $A,B \in \mathcal{R}$. m is subadditive if $m(A \cup B) \leq m(A) + m(B)$ for every $A,B \in \mathcal{R}$. $m_1 \leq m_2$ means $m_1(E) \leq m_2(E)$ for every $E \in \mathcal{R}$.

Standard references for results in measure and integration theory are Dinculeanu [12], Dunford and Schwartz [15], and Halmos [17]. We adhere to the notation of Dunford and Schwartz unless otherwise noted.

2. Properties of Set Functions. In this section we discuss some elementary properties of measures and their associated nonnegative set functions. $\mathcal{R}$ is a ring and $m:\mathcal{R} \to X$ is a measure unless otherwise noted. m is bounded if the set $m(\mathcal{R})$ is a bounded subset of X, that is, $\sup_{E\in\mathcal{R}} |m(E)| < \infty$. The total variation $v(m,E)$ of the measure m on a set $E \in \mathcal{R}$ is defined by

$$v(m,E) = \sup_{\pi(E)} \sum_{i=1}^{n} |m(E_i)|.$$

It is well known (see Dinculeanu [12]) that the set function $v(m)$ is finitely additive but may not be finite even though m is a bounded countably additive measure. As Theorem 2.2 (ii)

infra shows, $v(m)$ is bounded whenever m is a bounded scalar measure. For this reason $v(m)$ plays an important role in the theory of integration with respect to a scalar measure m, since it is a nonnegative measure which dominates m. When m is a countably additive vector measure and $v(m,E) < \infty$ for each $E \in \mathcal{R}$, Dinculeanu [12] has shown that most of the results from the scalar integration theory carry over to the vector case. In general, however, the total variation may be infinite and hence of little value. Thus we must use a more delicate device known as a control measure for m (see Theorem 3.3 *infra*). We now introduce a set function $\tilde{m}$ called the variation of m. For every $E \in \mathcal{R}$ we define

$$\tilde{m}(E) = \sup_{\substack{F \subseteq E \\ F \in \mathcal{R}}} |m(F)| \; .$$

It is obvious that m is bounded if and only if $\tilde{m}$ is bounded.

The following Lemma will be used several times in this dissertation.

2.1 Lemma. If $a_1,\ldots,a_n$ are scalars, then

$$\sum_{i=1}^{n} |a_i| \le 4 \sup_{\Delta \subseteq N_n} \Big| \sum_{i \in \Delta} a_i \Big| .$$

In particular, if $x_1,\ldots,x_n \in X$ and $x^* \in X^*$, then

$$\sum_{i=1}^{n} |x^*x_i| \le 4 \sup_{\Delta \subseteq N_n} \Big| \sum_{i \in \Delta} x^*x_i \Big| \le 4|x^*| \sup_{\Delta \subseteq N_n} \Big| \sum_{i \in \Delta} x_i \Big| ,$$

and
$$\Big| \sum_{i=1}^{n} a_i x_i \Big| \le 4 \sup_{1 \le i \le n} |a_i| \sup_{\Delta \subseteq N_n} \Big| \sum_{i \in \Delta} x_i \Big| .$$

Proof. Let $\Delta_1,\ldots,\Delta_4$ denote the sets of integers i such that $\mathrm{Re}\, a_i > 0$, $\mathrm{Re}\, a_i \leq 0$, $\mathrm{Im}\, a_i > 0$, and $\mathrm{Im}\, a_i \leq 0$ respectively. Then

$$\begin{aligned}\sum_{i=1}^{n} |a_i| &\leq \sum_{i=1}^{n} |\mathrm{Re}\, a_i| + \sum_{i=1}^{n} |\mathrm{Im}\, a_i| \\ &= \sum_{\Delta_1} \mathrm{Re}\, a_i - \sum_{\Delta_2} \mathrm{Re}\, a_i + \sum_{\Delta_3} \mathrm{Im}\, a_i - \sum_{\Delta_4} \mathrm{Im}\, a_i \\ &= \mathrm{Re} \sum_{\Delta_1} a_i - \mathrm{Re} \sum_{\Delta_2} a_i + \mathrm{Im} \sum_{\Delta_3} a_i - \mathrm{Im} \sum_{\Delta_4} a_i \\ &\leq 4 \sup_{\Delta \subseteq N_n} \Big|\sum_{i \in \Delta} a_i\Big|,\end{aligned}$$

where we have used the elementary inequalities $|a| \leq |\mathrm{Re}\, a| + |\mathrm{Im}\, a|$ and $|\mathrm{Re}\, a|$, $|\mathrm{Im}\, a| \leq |a|$ for any $a \in \Phi$. The inequalities concerning the x^*x_i's are then immediate. Since

$$\Big|\sum_{i=1}^{n} a_i x_i\Big| = \Big|x^* \sum_{i=1}^{n} a_i x_i\Big| \leq \sup_{i \leq i \leq n} |a_i| \sum_{i=1}^{n} |x^* x_i|$$

for some $x^* \in X_1^*$, the final inequality follows from the preceding one. □

As a first application of this lemma we establish the properties of the variation of a vector measure.

2.2 Theorem. (i) $\tilde{m}$ is nonnegative, monotone, and subadditive.

(ii) If m is scalar-valued, then $v(m) \leq 4\tilde{m}$.

(iii) $|m(E)| \leq \tilde{m}(E) \leq \sup_{X_1^*} v(x^*m, E) \leq 4\tilde{m}(E)$, for $E \in \mathcal{R}$.

Proof. (i) $\tilde{m}$ is clearly nonnegative and monotone by definition. If E, F, and G belong to $\mathcal{R}$ and $G \subseteq E \cup F$, then

$G = (G \cap E) + (G \cap (F \setminus E))$, so that

$$|m(G)| = |m(G \cap E) + m(G \cap (F \setminus E))|$$
$$\leq |m(G \cap E)| + |m(G \cap (F \setminus E))|$$
$$\leq \tilde{m}(E) + \tilde{m}(F).$$

Therefore $\tilde{m}(E \cup F) \leq \tilde{m}(E) + \tilde{m}(F)$ for every $E, F \in \mathcal{R}$.

(ii) If m is scalar-valued and $(E_i) \in \pi(E)$, then

$$\sum_{i=1}^{n} |m(E_i)| \leq 4 \sup_{\Delta \subseteq N_n} \left| \sum_{i \in \Delta} m(E_i) \right| \leq 4 \sup_{\substack{F \subseteq E \\ F \in \mathcal{R}}} |m(F)| ,$$

by Lemma 2.1. Thus $v(m,E) \leq 4\tilde{m}(E)$ for every $E \in \mathcal{R}$.

(iii) Suppose that $E \in \mathcal{R}$. It is obvious that $|m(E)| \leq \tilde{m}(E)$. Using (ii) we have

$$\tilde{m}(E) = \sup_{\substack{F \subseteq E \\ F \in \mathcal{R}}} |m(F)|$$
$$= \sup_{X_1^*} \sup_{\substack{F \subseteq E \\ F \in \mathcal{R}}} |x^*m(F)|$$
$$\leq \sup_{X_1^*} v(x^*m, E)$$
$$\leq 4 \sup_{X_1^*} \sup_{\substack{F \subseteq E \\ F \in \mathcal{R}}} |x^*m(F)|$$
$$= 4\tilde{m}(E). \square$$

2.3 <u>Remark</u>. By the principle of uniform boundedness (see Dunford and Schwartz [15]) any function with values in X is bounded if and only if it is weakly bounded. In particular, m is bounded if and only if x^*m is bounded for every $x^* \in X^*$. If $\mathcal{R}$ is a σ-ring and m is countably additive, then each x^*m is also countably additive. It is known (see Halmos [17])

that a countably additive scalar measure on a σ-ring is bounded. Hence m is also bounded. The Orlicz-Pettis Theorem (see Dunford and Schwartz [15]) states that, conversely, if x^*m is countably additive for each $x^* \in X^*$ and $\mathcal{R}$ is a σ-ring, then m is countably additive.

The hereditary ring $\mathcal{H}$ generated by a ring $\mathcal{R}$ is the family of all subsets of S which are contained in some element of $\mathcal{R}$. If $\mu:\mathcal{R} \to [0,\infty]$ is a monotone set function, then we can extend μ to $\mathcal{H}$ as follows. Define

$$\mu^1(E) = \inf_{\substack{F \supseteq E \\ F \in \mathcal{R}}} \mu(F),$$

for every $E \in \mathcal{H}$. It is immediate that μ^1 is a monotone extension of μ, and $\mu^1 \geq 0$. If μ is subadditive then so is μ^1. To see this, suppose that $G,H \in \mathcal{H}$ and $\varepsilon > 0$ is fixed. Choose $E,F \in \mathcal{R}$ such that $G \subseteq E$, $H \subseteq F$, and $\mu(E) < \mu^1(G) + \varepsilon$, $\mu(F) < \mu^1(H) + \varepsilon$. Then $G \cup H \subseteq E \cup F$ and so

$$\mu^1(G \cup H) \leq \mu(E \cup F) \leq \mu(E) + \mu(F) < \mu^1(G) + \mu^1(H) + 2\varepsilon.$$

We conclude that $\mu^1(G \cup H) \leq \mu^1(G) + \mu^1(H)$. In this way we can extend $\tilde{m}$ to $\mathcal{H}$. A set $A \in \mathcal{H}$ is said to be an m-null set, or simply a null set when m is understood, if $\tilde{m}(A) = 0$. We shall always identify the extension of $\tilde{m}$ to $\mathcal{H}$ by $\tilde{m}$ unless there is a possibility of confusion. Note that A is a null set if and only if for every $\varepsilon > 0$, there is an $E \in \mathcal{R}$ such that $A \subseteq E$ and $\tilde{m}(E) < \varepsilon$.

3. <u>Strongly Bounded Measures</u>. We now introduce a fundamental property of certain vector measures, and discuss some of its implications. $\mathcal{R}$ is a ring and $m:\mathcal{R} \to X$ is a measure unless otherwise noted.

3.1 Definitions. (i) m is said to be strongly bounded (s-bounded) if for every disjoint sequence (E_i) in $\mathcal{R}$, $\lim m(E_i) = 0$.

This property was introduced by Rickart [25], and later used by Brooks and Jewett [6] to establish convergence theorems for vector measures. Brooks [4] showed that s-boundedness is equivalent to the existence of a control measure. This in turn provides necessary and sufficient conditions for the existence of an extension of a countably additive measure from Σ to $\sigma(\Sigma)$ when Σ is an algebra. These results are presented in Theorems 3.3 and 3.4 *infra*.

(ii) A series Σx_n in X is unconditionally convergent if for every $\varepsilon > 0$ there is a Δ_ε such that $\Delta \cap \Delta_\varepsilon = \emptyset$ implies that $|\sum_{i\in\Delta} x_i| < \varepsilon$.

The following result states some basic properties of an s-bounded measure. The proof is omitted.

3.2 Theorem. (Rickart [25]) (i) If m is s-bounded then m is bounded.

(ii) The following statements are equivalent:

(a) m is s-bounded.

(b) For every disjoint sequence (E_i) in $\mathcal{R}$ we have $\lim \tilde{m}(E_i) = 0$.

(c) For every disjoint sequence (E_i) in $\mathcal{R}$, $\Sigma m(E_i)$ is unconditionally convergent.

Suppose that $\mathfrak{m}$ is a family of measures on $\mathcal{R}$. We say that the measures in $\mathfrak{m}$ are uniformly m-continuous, or uniformly continuous with respect to m, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\mu(A)| < \varepsilon$ for every $\mu \in \mathfrak{m}$ if $\tilde{m}(A) < \delta$. When $\mathfrak{m}$ contains only one measure μ we say that μ is m-continuous. μ and m are mutually continuous if each is continuous with respect to the other.

The most important characterization of s-bounded measures is the following result due to Brooks [4] which we state without proof.

3.3 <u>Theorem</u>. (Brooks) $m:\mathcal{R} \to X$ is s-bounded if and only if there is a bounded nonnegative measure $\underline{m}$ on $\mathcal{R}$ such that

(i) m is $\underline{m}$-continuous.

(ii) $\underline{m} \leq \tilde{m}$ on $\mathcal{R}$.

Moreover, m is countably additive if and only if $\underline{m}$ is countably additive.

The measure $\underline{m}$ is called a control measure for m. (i) and (ii) show that m and $\underline{m}$ are mutually continuous; it follows that m and $\underline{m}$ have the same class of null sets.

A fundamental problem in measure theory is to find conditions under which a countably additive measure on a ring $\mathcal{R}$ has a countably additive extension to the σ-ring generated by $\mathcal{R}$. Using the previous result of Brooks we can prove an extension theorem for countably additive s-bounded measures on an algebra. This subject has been investigated by many authors. Takahashi [28] introduced a kind of boundedness

condition less natural than, but equivalent to, the concept of s-boundedness. His result in the case of an algebra is therefore included in the following theorem. A measure $m:\mathcal{R} \to X$ is said to be T-bounded if for every $\varepsilon > 0$ there is an $n \in N$ such that if $E_1,\ldots,E_n$ are disjoint sets in $\mathcal{R}$, then $|m(E_i)| < \varepsilon$ for some $i \in N_n$.

To see that T-boundedness is equivalent to s-boundedness, we remark that if m is not s-bounded then there is a disjoint sequence (E_i) and an $\varepsilon > 0$ such that $|m(E_i)| > \varepsilon$, $i \in N$, so m cannot be T-bounded. This shows that if m is T-bounded then m is s-bounded. Conversely, suppose that m is s-bounded with control measure $\underline{m}$ as in Theorem 3.3. Since $\underline{m}$ is a bounded nonnegative measure there is a constant K such that $\sum_{i=1}^{n} \underline{m}(E_i) \leq K$ for every finite disjoint family (E_i). If $\underline{m}$ is not T-bounded, then there is an $\varepsilon > 0$ such that for every $n \in N$ there are disjoint sets $E_{1n},\ldots,E_{nn}$ in $\mathcal{R}$ with $\underline{m}(E_{in}) > \varepsilon$, $1 \leq i \leq n$. This clearly contradicts the boundedness of $\underline{m}$, so $\underline{m}$ is T-bounded. Since m and $\underline{m}$ are mutually continuous, m is also T-bounded.

We now state the extension theorem for measures on an algebra.

3.4 <u>Theorem</u>. (Brooks [5]) Suppose Σ is an algebra and $m:\Sigma \to X$ is countably additive and s-bounded. Then m has a unique countably additive extension to $\sigma(\Sigma)$.

<u>Remark</u>. Conversely, if m has a countably additive extension to $\sigma(\Sigma)$, then m is s-bounded on Σ.

Proof. Let $\underline{m}$ be a control measure for m as in Theorem 3.3. Then $\underline{m}$ is countably additive and bounded on Σ. It is well known (see Halmos [17]) that a nonnegative, bounded, countably additive measure on an algebra has a unique extension to a bounded, countably additive measure on $\sigma(\Sigma) = \Sigma_1$. We identify this extension by $\underline{m}$.

The symmetric difference of two sets A and B, denoted by $A \Delta B$, is defined by $A \Delta B = (A\backslash B) + (B\backslash A)$. We also have $A \Delta B = (A \cup B)\backslash(A \cap B)$. The elementary set relations

(i) $A \Delta C \subseteq A \Delta B \cup B \Delta C$,

(ii) $A \cup B \Delta C \cup D \subseteq A \Delta C \cup B \Delta D$,

(iii) $A \cap B \Delta C \cap D \subseteq A \Delta C \cup B \Delta D$,

(iv) $A\backslash B \Delta C\backslash D \subseteq A \Delta C \cup B \Delta D$,

are straightforward to verify. Define $d:\Sigma_1 \times \Sigma_1 \to [0,\infty)$ by $d(A,B) = \underline{m}(A \Delta B)$. Since $\underline{m}$ is subadditive, (i) shows that d satisfies the triangle inequality, and hence is a semi-metric for Σ_1. Moreover, from the $\underline{m}$-continuity of m on Σ and from the inequalities $\underline{m}(A\backslash B) \leq \underline{m}(A \Delta B)$ and $|m(A) - m(B)| \leq |m(A\backslash B)| + |m(B\backslash A)|$, it follows that m is a uniformly continuous mapping from Σ into X.

Since Σ_1 is the smallest σ-algebra containing Σ, it is known that for every $A \in \Sigma_1$ there is a sequence (A_n) in Σ such that $\lim \underline{m}(A_n \Delta A) = 0$ (see Halmos [17]). This fact shows that Σ is a dense subset of Σ_1. If we identify sets A and B in Σ_1 whenever $d(A,B) = 0$, then m is well-defined on Σ, since $m(A) = m(B)$ whenever $d(A,B) = 0$ for $A,B \in \Sigma$. Since X is complete, m has a unique extension to a continuous mapping from Σ_1 into X. We denote this extension by m.

To see that m is finitely additive on Σ_1, we remark that (ii), (iii), and (iv) show that the mappings $(A,B) \to A \cup B$, $A \cap B$, and $A \backslash B$ respectively from $\Sigma_1 \times \Sigma_1 \to \Sigma_1$ are continuous. Suppose $A,B \in \Sigma_1$ and $A \cap B = \emptyset$. Let (A_n) and (B_n) be sequences in Σ converging to A and B respectively. (ii) and (iv) imply that the mapping $(A,B) \to A \Delta B$ is also a continuous function, so $\lim d(A_n \Delta B_n, A \Delta B) = 0$. By the continuity of m we have $\lim m(A_n \Delta B_n) = m(A \Delta B) = m(A + B)$ since $A \Delta B = A + B$. But $m(A_n \Delta B_n) = m(A_n \backslash B_n) + m(B_n \backslash A_n)$, and by the continuity of $(A,B) \to A\backslash B$ we have $\lim m(A_n \backslash B_n) = m(A \backslash B) = m(A)$ and $\lim m(B_n \backslash A_n) = m(B)$. Thus $m(A + B) = m(A) + m(B)$.

Finally, to see that m is countably additive on Σ_1 it suffices to show that if $A_i \searrow \emptyset$ then $\lim m(A_i) = 0$. Since $\underline{m}$ is countably additive, $\lim \underline{m}(A_i) = 0$. This in turn implies that $\lim m(A_i) = 0$ since m is continuous on the space (Σ_1, d). □

We quote without proof a final result from the theory of vector measures. The following theorem is a generalization of the well-known Vitali-Hahn-Saks Theorem.

3.5 Theorem. (Brooks and Jewett [6]). Suppose $\mathcal{R}$ is a σ-ring and $\nu : \mathcal{R} \to [0,\infty)$ is a bounded measure. If $\mu_n : \mathcal{R} \to X$ is a ν-continuous measure for each n, and if $\lim \mu_n(E)$ exists for each $E \in \mathcal{R}$, then the μ_n are uniformly ν-continuous.

4. Convergence in Measure and the Space $M(m)$. This section introduces the concept of convergence in measure for sequences of functions. Using this concept we can define the class of

functions with which the integration theory of Section 5 will be concerned. From now on Σ will denote an algebra and $m:\Sigma \to X$ is a bounded measure. Then $\tilde{m}$ is a nonnegative, monotone, subadditive, bounded set function, and, in fact, these are the only properties that are needed to obtain the results of this section. $F(S)$ denotes the family of all scalar functions on S. If $f \in F(S)$ we use the abbreviation $[|f| > \epsilon]$ for the set $\{s \in S: |f(s)| > \epsilon\}$.

4.1 Definition. A sequence (f_k) in $F(S)$ is Cauchy in measure if for every $\epsilon > 0$ we have

$$\lim_{i,j\to\infty} \tilde{m}[|f_i - f_j| > \epsilon] = 0.$$

A sequence (f_k) in $F(S)$ converges to $f \in F(S)$ in measure if for every $\epsilon > 0$ we have

$$\lim_{k\to\infty} \tilde{m}[|f_k - f| > \epsilon] = 0.$$

If (f_k) converges to f in measure we write $m\text{-}\lim f_k = f$. The following lemma gives a useful equivalent formulation of convergence in measure.

4.2 Lemma. Suppose $f, f_k \in F(S)$, $k \in N$. Then $m\text{-}\lim f_k = f$ if and only if there are sets $A_k \in \Sigma$ and a sequence of positive numbers ϵ_k converging to zero such that $\lim \tilde{m}(A_k) = 0$ and $|f_k(s) - f(s)| \le \epsilon_k$ if $s \notin A_k$.

Proof. Let (A_k) and (ϵ_k) satisfy the stated properties, and suppose $\epsilon, \delta > 0$ are given. Choose n such that $\epsilon_k < \epsilon$ and $\tilde{m}(A_k) < \delta$ if $k \ge n$. Since $[|f_k - f| > \epsilon] \subseteq [|f_k - f| > \epsilon_k] \subseteq A_k$ if $k \ge n$, we conclude that $\tilde{m}[|f_k - f| > \epsilon] < \delta$ if $k \ge n$. Thus $m\text{-}\lim f_k = f$.

Conversely, suppose $m - \lim f_k = f$. For $f \in F(S)$, Dunford and Schwartz [15] define the quantity

$$\|f\| = \inf_{\epsilon>0} (\epsilon + \tilde{m}[|f| > \epsilon]),$$

which is finite since m is bounded. To see that $\lim\|f_k - f\| = 0$, suppose, to the contrary, that $\lim\|f_k - f\| > 0$. Without loss of generality we may suppose that $\|f_k - f\| > \epsilon$ for $k \in N$ and some $\epsilon > 0$. There is an n such that $\tilde{m}[|f_k - f| > \epsilon/2] < \epsilon/2$ if $k \geq n$. Hence

$$\epsilon < \|f_k - f\| \leq \epsilon/2 + \tilde{m}[|f_k - f| > \epsilon/2] < \epsilon,$$

which is a contradiction. Therefore $\lim\|f_k - f\| = 0$. For each $k \in N$ choose $\epsilon_k > 0$ such that $\epsilon_k + \tilde{m}[|f_k - f| > \epsilon_k] <$ $< \|f_k - f\| + 1/k$. Then choose $A_k \in \Sigma$ such that $[|f_k - f| > \epsilon_k] \subsetneq A_k$ and $\tilde{m}(A_k) < \tilde{m}[|f_k - f| > \epsilon_k] + 1/k$. The sequences (A_k) and (ϵ_k) evidently satisfy the properties stated in the lemma. □

If $E \subseteq S$, then χ_E denotes the indicator function of E, that is, the function whose value is 1 for $s \in E$ and 0 for $s \notin E$. We now define an important class of functions.

4.3 Definition. A Σ-simple function is any function $f \in F(S)$ which can be represented in the form

$$f = \sum_{i=1}^{n} a_i \chi_{E_i},$$

where $a_i \in \Phi$, and the sets $E_i \in \Sigma$ are disjoint and satisfy $S = \Sigma E_i$. When Σ is understood, f is called a simple function.

Note that every simple function f has a canonical representation with $E_i = [f = a_i]$.

4.4 Definition. $f \in F(S)$ is measurable if there is a sequence of simple functions converging to f in measure.

For $f \in F(S)$ and $E \subseteq S$ we define the oscillation of f on E, written $O(f,E)$, by

$$O(f,E) = \sup_{s,t \in E} |f(s) - f(t)|.$$

We have the following characterization of measurability.

4.5 Theorem. f is measurable if and only if for every $\varepsilon > 0$ there is a partition $(E_i) \in \pi(S)$ such that $\tilde{m}(E_1) < \varepsilon$ and $O(f,E_i) < \varepsilon$ for $2 \le i \le n$.

Proof. Suppose (f_k) is a sequence of simple functions converging to f in measure. Let (A_k) and (ε_k) satisfy the conditions of Lemma 4.2 for f and (f_k). Fix $\varepsilon > 0$, and choose k such that $\tilde{m}(A_k) < \varepsilon$ and $\varepsilon_k < \varepsilon/3$. Suppose that $f_k = \sum_{i=1}^{n} a_i \chi_{E_i}$; define a partition of S by setting $F_1 = A_k$ and $F_i = E_{i-1} \setminus A_k$ for $2 \le i \le n+1$. If $s,t \in F_i$ for some $i > 1$, then $|f(s) - f(t)| \le |f(s) - f_k(s)| + |f_k(t) - f(t)| \le 2\varepsilon/3$. Hence $O(f,F_i) < \varepsilon$ for $2 \le i \le n+1$, so (F_i) is the desired partition.

Conversely, if the conditions of the theorem hold then choose a sequence of partitions P_k for $\varepsilon = 1/k$. If $P_k = (E_{ik})$, set

$$f_k = \sum_{i=1}^{n_k} f(s_{ik}) \chi_{E_{ik}},$$

where $s_{ik} \in E_{ik}$ for $1 \le i \le n_k$. If $s \notin E_{1k}$ then $|f(s) - f_k(s)| \le \max_{2 \le i \le n_k} O(f,E_i) \le 1/k$. Since $\tilde{m}(E_{1k}) < 1/k$, the sequences (E_{1k}) and $(1/k)$ satisfy the conditions of Lemma 4.2, and so $m\text{-}\lim f_k = f$. We conclude that f is measurable. □

Let $M(m) = M$ denote the space of all measurable functions in $F(S)$. Since we have introduced a concept of convergence in $F(S)$ we can discuss the topological properties of M. The following result from Dunford and Schwartz [15] lists some of the useful properties of the space M. We shall omit the proof.

4.6 Theorem. (Dunford and Schwartz) M is a closed linear algebra in $F(S)$. If $g:\Phi \to \Phi$ is continuous, then the mapping $f \to g \circ f$ is a continuous function from M into M.

The following result will be very useful in connection with the Dominated Convergence Theorem.

4.7 Theorem. If (f_k) is a sequence of simple functions and $m\text{-}\lim f_k = f$, then there is a sequence (g_k) of simple functions such that $m\text{-}\lim g_k = f$, and $|g_k| \leq 2|f|$ on S, for each $k \in N$.

Proof. Let (A_k) and (ε_k) be sequences as in Lemma 4.2 for f and (f_k). Define $g_k(s) = f_k(s)$ if $s \notin A_k$ and $|f_k(s)| > 2\varepsilon_k$, and $g_k(s) = 0$ otherwise, for each k. Then if $s \notin A_k$ and $|f_k(s)| > 2\varepsilon_k$, we have $|g_k(s) - f(s)| < \varepsilon_k$. If $s \notin A_k$ and $|f_k(s)| \leq 2\varepsilon_k$ then

$$|f(s)| \leq |f(s) - f_k(s)| + |f_k(s)| \leq 3\varepsilon_k,$$

so $|f(s) - g_k(s)| \leq 3\varepsilon_k$ if $s \notin A_k$. By Lemma 4.2 we conclude that $m\text{-}\lim g_k = f$. If $s \in A_k$ or if $|f_k(s)| \leq 2\varepsilon_k$, then $g_k(s) = 0$ so certainly $|g_k(s)| \leq 2|f(s)|$. If $s \notin A_k$ and $|f_k(s)| > 2\varepsilon_k$, then

$$
\begin{aligned}
|f(s)| &\geq |f_k(s)| - |f_k(s) - f(s)| \\
&\geq |f_k(s)| - \varepsilon_k \\
&\geq 1/2|f_k(s)| = 1/2|g_k(s)|.
\end{aligned}
$$

Thus $|g_k(s)| \leq 2|f(s)|$ for $s \in S$ and $k \in N$. □

5. A Lebesgue-type Integral. We now present the standard results of integration theory. Many of these theorems were stated by Bartle [1] in his fundamental paper; they are included for completeness. Throughout this section Σ is an algebra and $m:\Sigma \to X$ is a bounded measure.

If f is a simple function with representation $\sum_{i=1}^{n} a_i \chi_{E_i}$, we define the integral of f over $E \in \Sigma$ by

$$\int_E f dm = \sum_{i=1}^{n} a_i m(E \cap E_i).$$

The following result lists the basic properties of the integral for simple functions. The set function $\lambda(E) = \int_E f dm$ is called the indefinite integral of f.

5.1 Theorem. (Bartle) (i) The integral of a simple function is independent of the function's representation.

(ii) For a fixed set $E \in \Sigma$ the integral over E is a linear map from the linear space of simple functions into X.

(iii) For a fixed simple function f the indefinite integral of f is a measure on Σ.

(iv) If f is a simple function bounded by a constant K on a set E, then

$$\left|\int_E f dm\right| \leq 4K\tilde{m}(E).$$

(v) If $U \in B(X,Y)$ and f is a simple function, then for every $E \in \Sigma$,

$$\int_E f\,dUm = U\left(\int_E f\,dm\right).$$

Note that by (iv) the indefinite integral of a simple function is a bounded and m-continuous measure. If m is s-bounded, then so is the indefinite integral of a simple function.

The Bartle integral is defined using sequences of simple functions. The next two theorems discuss the properties of these sequences which enable us to construct the general integral. Bartle proved only that (i) implies (ii) in Theorem 5.2.

5.2 <u>Theorem</u>. Suppose (f_k) is a sequence of simple functions that is Cauchy in measure. Then the following statements are equivalent:

(i) The indefinite integrals of the f_k's are uniformly m-continuous.

(ii) $\lim \int_E f_k\,dm$ exists uniformly for $E \in \Sigma$.

<u>Proof</u>. Suppose (i) holds and $\varepsilon > 0$ is given. Choose $\delta > 0$ such that $E \in \Sigma$ and $\tilde{m}(E) < \delta$ implies that for each $k \in N$,

$$\left|\int_E f\,dm\right| < \varepsilon.$$

There is an $n \in N$ and sets $A_{ij} \in \Sigma$ such that $\tilde{m}(A_{ij}) < \delta$ and $|f_i(s) - f_j(s)| \le \varepsilon/4\tilde{m}(S)$, if $s \notin A_{ij}$ and $i,j \ge n$. Then for $E \in \Sigma$ and $i,j \ge n$, we have

$$\begin{aligned}\left|\int_E f_i\,dm - \int_E f_j\,dm\right| &\le \left|\int_{E\setminus A_{ij}} (f_i - f_j)\,dm\right| + \\ &+ \left|\int_{E\cap A_{ij}} f_i\,dm\right| + \left|\int_{E\cap A_{ij}} f_j\,dm\right| \\ &\le [4\varepsilon/4\tilde{m}(S)]\tilde{m}(E\setminus A_{ij}) + 2\varepsilon \le 3\varepsilon,\end{aligned}$$

using (iii) and (iv) of Theorem 5.1. Since E was arbitrary, we conclude that the sequences $(\int_E f_k dm)$ are Cauchy uniformly for $E \in \Sigma$. Since X is complete, (ii) holds.

Conversely, if (ii) holds and $\varepsilon > 0$ is given, then there is an $n \in N$ such that for every $E \in \Sigma$,

$$|\int_E (f_k - f_n) dm| < \varepsilon,$$

provided $k \geq n$. Since each indefinite integral is m-continuous, there is a $\delta > 0$ such that if $A \in \Sigma$ and $\tilde{m}(A) < \delta$ then

$$|\int_A f_k dm| < \varepsilon,$$

for $1 \leq k \leq n$. If $\tilde{m}(A) < \delta$ and $k \geq n$, we also have

$$\begin{aligned} |\int_A f_k dm| &\leq |\int_A (f_k - f_n) dm| + |\int_A f_n dm| \\ &\leq 2\varepsilon. \end{aligned}$$

Therefore the indefinite integrals of the f_k's are uniformly m-continuous. □

5.3 <u>Theorem</u>. Suppose (f_k) and (g_k) are sequences of simple functions such that

(i) $\text{m-lim}(f_k - g_k) = 0$.

(ii) The indefinite integrals of the f_k's and the g_k's are uniformly m-continuous. Then

$$\lim \int_E f_k dm = \lim \int_E g_k dm$$

uniformly for $E \in \Sigma$.

<u>Proof</u>. Since $h_k = f_k - g_k$ is a simple function for each k, and since

$$|\int_E h_k dm| \leq |\int_E f_k dm| + |\int_E g_k dm|$$

for every $E \in \Sigma$, Theorem 5.2 implies that $\lim \int_E h_k dm$ exists for every $E \in \Sigma$. We need only show that this limit is zero

uniformly in E. Fix $\varepsilon > 0$. There is a $\delta > 0$ such that if $A \in \Sigma$ and $\tilde{m}(A) < \delta$, then for each $k \in N$,

$$|\int_A h_k dm| < \varepsilon.$$

Since m-lim $h_k = 0$, there is an $n \in N$ and sets $A_k \in \Sigma$ such that $\tilde{m}(A_k) < \delta$ and $|h_k(s)| < \varepsilon/4\tilde{m}(S)$ if $s \notin A_k$, provided $k \geq n$. Thus for $E \in \Sigma$ and $k \geq n$,

$$\begin{aligned} |\int_E h_k dm| &\leq |\int_{E\setminus A_k} h_k dm| + |\int_{E\cap A_k} hdm| \\ &\leq [4\varepsilon/4\tilde{m}(S)]\tilde{m}(E\setminus A_k) + \varepsilon \\ &\leq 2\varepsilon. \end{aligned}$$

That is, $\lim \int_E h_k dm = 0$ uniformly for $E \in \Sigma$. □

Following Bartle [1] we now define the general integral.

5.4 <u>Definition</u>. $f \in F(S)$ is integrable if there is a sequence (f_k) of simple functions such that

(i) m-lim $f_k = f$.

(ii) The indefinite integrals of the f_k's are uniformly m-continuous.

If f is integrable then any sequence (f_k) of simple functions satisfying (i) and (ii) of Definition 5.4 is said to determine f. Theorem 5.2 shows that if (f_k) determines f, then $\lim \int_E f_k dm$ exists uniformly in E. We denote this limit by the usual symbols

$$\int_E f dm \qquad \text{or} \qquad \int_E f(s)m(ds).$$

Theorem 5.3 shows that the integral of f is independent of the determining sequence. Let L(m) denote the family of all functions that are integrable with respect to m. The following

theorem lists some standard properties of the integral. We omit the proof.

5.5 Theorem. (i) For a fixed set $E \in \Sigma$, the integral over E is a linear mapping from the linear space $L(m)$ into X.

(ii) For a fixed function $f \in L(m)$, the indefinite integral of f is an m-continuous measure on Σ.

Suppose $f \in M$ and there is a null set A such that $\sup_{s \notin A} |f(s)| < \infty$. Then we say that f is essentially bounded, and define the essential supremum $\|f\|$ of f by

$$\|f\| = \inf_{\tilde{m}(A)=0} \ \sup_{s \notin A} |f(s)|.$$

The following standard result, whose proof refines that of Bartle, shows that every measurable and essentially bounded function is integrable.

5.6 Theorem. If $f \in M$ is essentially bounded, then $f \in L(m)$ and for every $E \in \Sigma$ we have

$$\left|\int_E f\,dm\right| \leq 4\|f\|\tilde{m}(E).$$

Proof. Let $K = \|f\|$, and suppose (f_k) is a sequence of simple functions converging to f in measure. Define

$$g_k(s) = \begin{cases} f_k(s) & |f_k(s)| \leq K, \\ (K/|f_k(s)|)f_k(s) & |f_k(s)| > K, \end{cases}$$

for each $s \in S$ and $k \in N$. Then (g_k) is a sequence of simple functions, and

$$|g_k(s) - f_k(s)| = \begin{cases} |f_k(s)| - K & |f_k(s)| > K \\ 0 & |f_k(s)| \leq K, \end{cases}$$

since $|g_k(s) - f_k(s)| = |f_k(s)(1-K/|f_k(s)|)| = |f_k(s)| - K$ if $|f_k(s)| > K$. Suppose $\epsilon > 0$. Since f is essentially bounded by K, there is a null set A such that $|f(s)| \le K + \epsilon/4$ if $s \notin A$. Now if $s \notin A$ and $|f_k(s)| > K + \epsilon/2$, then since $|f(s)| \le K + \epsilon/4$, we have

$$\begin{aligned} |f_k(s) - f(s)| &\ge |f_k(s)| - |f(s)| \\ &> K + \epsilon/2 - K - \epsilon/4 = \epsilon/4. \end{aligned}$$

Therefore
$$\begin{aligned} [|g_k - f_k| > \epsilon/2] &= [|f_k| > K + \epsilon/2] \\ &\subseteq A \cup [|f_k - f| > \epsilon/4]. \end{aligned}$$

Finally,

$$\begin{aligned} [|g_k - f| > \epsilon] &\subseteq [|g_k - f_k| > 3\epsilon/4] \cup [|f_k - f| > \epsilon/4] \\ &\subseteq A \cup [|f_k - f| > \epsilon/4]. \end{aligned}$$

Since A is null and m-lim $f_k = f$, we conclude that m-lim $g_k = f$. Since the g_k's are uniformly bounded by K, Theorem 5.1 (iv) implies that for every k,

$$|\int_E g_k dm| \le 4K\tilde{m}(E)$$

for any $E \in \Sigma$. This shows that the indefinite integrals of the g_k's are uniformly m-continuous. It follows that $f \in L(m)$, and that for every $E \in \Sigma$,

$$|\int_E f dm| = \lim |\int_E g_k dm| \le 4K\tilde{m}(E). \square$$

Using Theorem 5.6 and the m-continuity of the indefinite integral of a function $f \in L(m)$ we have the following Corollary.

5.7 <u>Corollary</u>. If $f \in L(m)$, then the indefinite integral of f is a bounded measure on Σ.

Proof. Since f is measurable, f is bounded except possibly on sets of arbitrarily small variation. Choose $\delta > 0$ such that if $\tilde{m}(E) < \delta$, then

$$\left|\int_E f dm\right| < 1.$$

Choose $E \in \Sigma$ such that $\tilde{m}(E) < \delta$ and f is bounded on $S \setminus E$, say by K. For any $A \in \Sigma$, therefore

$$\left|\int_A f dm\right| \leq \left|\int_{A\setminus E} f dm\right| + \left|\int_{A\cap E} f dm\right|$$
$$\leq 4K\tilde{m}(S) + 1. \square$$

When Σ is a σ-algebra and m is a countably additive vector measure, Bartle, Dunford and Schwartz [2] have shown that the integral satisfies the following property. If $f \in L(m)$ and $U \in B(X,Y)$, then $f \in L(Um)$ and for each $E \in \Sigma$,

$$\int_E f dUm = U\left(\int_E f dm\right).$$

We now extend this result to the case where m is a bounded measure on an algebra Σ.

5.8 Theorem. Suppose that $U \in B(X,Y)$. If $f \in L(m)$ then $f \in L(Um)$, and for every $E \in \Sigma$ we have

$$\int_E f dUm = U\left(\int_E f dm\right).$$

Proof. Since $|Um(E)| \leq |U||m(E)|$, it follows at once that $|Um(E)| \leq |U|\tilde{m}(E)$ for every $E \in \Sigma$. From this inequality we see that if m-lim $f_k = f$, then Um - lim $f_k = f$ as well. If $f \in L(m)$ and (f_k) is a sequence of simple functions determining f, then we need only show that the indefinite integrals of the f_k's with respect to Um are uniformly Um-continuous. Since we have

$$\left|\int_E (f_k - f_j)\,dUm\right| \le |U|\left|\int_E (f_k - f_j)\,dm\right|,$$

using Theorem 5.1 (v), this follows immediately from Theorem 5.2 and the fact that $\lim \int_E f_k dm$ exists uniformly for $E \in \Sigma$. □

When m is a scalar measure, Dunford and Schwartz [15] have introduced the following definition of integrability. $f \in F(S)$ is integrable if there is a sequence (f_k) of simple functions converging to f in measure, such that

$$\lim_{i,j\to\infty} \int_S |f_i - f_j|\,dv(m) = 0.$$

We shall show that the Bartle concept of integration and that of Dunford and Schwartz coincide when m is a bounded scalar measure. We state the following result without proof.

5.9 Theorem. (Dunford and Schwartz [15]) If m is a scalar measure and f is integrable (in the sense of Dunford and Schwartz), then the total variation of the indefinite integral $\lambda(E) = \int_E f dm$ is given by

$$v(\lambda,E) = \int_E |f(s)|\,v(m,ds).$$

Using Theorem 5.9 we now prove the equivalence of the Bartle and the Dunford and Schwartz integration theories in the case of a scalar measure.

5.10 Theorem. Suppose m is a bounded scalar measure. Then $f \in L(m)$ if and only if f is integrable in the sense of Dunford and Schwartz. In this case the same sequence of simple functions determines f for both definitions; consequently the two integrals coincide.

Proof. Suppose (f_k) is a sequence of simple functions. By Theorems 5.9 and 2.2 (ii) we have

$$\left|\int_E f_i dm - \int_E f_j dm\right| \le \int_E |f_i - f_j| dv(m)$$
$$\le 4 \sup_{\substack{F \subseteq E \\ F \in \Sigma}} \left|\int_F (f_i - f_j) dm\right|$$

for every $E \in \Sigma$. Suppose $f \in L(m)$ and (f_k) is a sequence determining f. Then by Theorem 5.2 the right-hand term in the inequality goes to zero uniformly for $E \in \Sigma$ as $i, j \to \infty$. Hence

$$(*) \qquad \lim_{i,j\to\infty} \int_S |f_i - f_j| dv(m) = 0,$$

and so f is integrable in the sense of Dunford and Schwartz. Conversely, if $m - \lim f_k = f$ and (*) holds, then the left-hand term of the inequality converges to zero uniformly for $E \in \Sigma$ as $i, j \to \infty$. This shows that the sequences $(\int_E f_k dm)$ are Cauchy uniformly in E, and hence by the completeness of X, they converge uniformly in E. By Theorem 5.2 we conclude that the indefinite integrals of the f_k's are uniformly m-continuous, so $f \in L(m)$. In either case, the same sequence of simple functions determines f for both definitions, so the two integrals coincide. □

We now prove some results concerning the domination of functions by integrable functions.

5.11 Lemma. Suppose (f_k) is a sequence of integrable functions, and $g \in L(m)$. If $|f_k| \le |g|$ on S for each k, then

$$\left|\int_E f_k dm\right| \le 4 \sup_{\substack{F \subseteq E \\ F \in \Sigma}} \left|\int_F g dm\right|$$

for each $E \in \Sigma$; consequently the indefinite integrals of the f_k's are uniformly m-continuous.

Proof. We use Theorem 5.8 with $U = x^* \in X^*$, and also Theorems 5.10, 5.9, and 2.2 (ii) to compute

$$\begin{aligned}
\left|\int_E f_k dm\right| &= \sup_{X_1^*} \left|\int_E f_k dx^*m\right| \\
&\le \sup_{X_1^*} \int_E |f_k| dv(x^*m) \\
&\le \sup_{X_1^*} \int_E |g| dv(x^*m) \\
&\le 4 \sup_{\substack{F\subseteq E \\ F\in\Sigma}} \sup_{X_1^*} \left|\int_F g dx^*m\right| \\
&= 4 \sup_{\substack{F\subseteq E \\ F\in\Sigma}} \left|\int_F g dm\right|,
\end{aligned}$$

for any $k \in N$. Since the indefinite integral of g is m-continuous, it follows that the indefinite integrals of the f_k's are uniformly m-continuous. □

5.12 Theorem. If $f \in M$, $g \in L(m)$, and $|f| \le |g|$ on S, then $f \in L(m)$.

Proof. Since $f \in M$ there is a sequence (f_k) of simple functions converging to f in measure. By Theorem 4.7 we may assume that $|f_k| \le 2|f|$ on S for every k. Then $|f_k| \le 2|g|$ on S, and $2g \in L(m)$. By Lemma 5.11 we conclude that the indefinite integrals of the f_k's are uniformly m-continuous. Therefore $f \in L(m)$. □

5.13 Corollary. Suppose $f \in M$. Then $f \in L(m)$ if and only if $|f| \in L(m)$. If (f_k) is a sequence of simple functions deter-

mining f, then $(|f_k|)$ is a sequence of simple functions determining $|f|$.

Proof. The first statement follows from Theorem 5.12. If f_k is a simple function, then so is $|f_k|$, and by Lemma 5.11,

$$\left|\int_E |f_k| dm\right| \le 4 \sup_{\substack{F \subseteq E \\ F \in \Sigma}} \left|\int_F f_k dm\right|.$$

Thus the indefinite integrals of the $|f_k|$'s are uniformly m-continuous, if the indefinite integrals of the f_k's are. Finally, since $||f_k| - |f|| \le |f_k - f|$, it follows that $[||f_k| - |f|| > \varepsilon] \subseteq [|f_k - f| > \varepsilon]$. If (f_k) determines f, we conclude that $(|f_k|)$ determines $|f|$. □

6. Convergence Theorems. In this section we prove stronger forms of the convergence theorems for sequences of integrals stated by Bartle [1]. Suppose that f and (f_k) are integrable functions. We say that f and (f_k) satisfy property B if

$$\lim \int_E f_k dm = \int_E f dm$$

uniformly for $E \in \Sigma$.

The main convergence theorem resembles the classical Vitali convergence theorem.

6.1 Theorem. Suppose $f \in F(S)$ and $(f_k) \subseteq L(m)$. Then $f \in L(m)$ and $f, (f_k)$ satisfy property B if and only if

(i) $m - \lim f_k = f$.

(ii) The indefinite integrals of the f_k's are uniformly m-continuous.

Proof. Suppose $f \in L(m)$ and $f, (f_k)$ satisfy property B. To prove that $m - \lim f_k = f$ we proceed as follows. By Theorem 5.13, $g_k = |f - f_k| \in L(m)$ for each $k \in N$. Let (h_{kj}) be a sequence of simple functions determining g_k as in Definition 5.4. By Theorem 5.13, $(|h_{kj}|)$ is a sequence of nonnegative simple functions determining $|g_k| = g_k$, so without loss of generality we may assume that $h_{kj} \geq 0$ for each $j \in N$.

By Theorem 5.8 we have $g_k \in L(x^*m)$ for every $x^* \in X^*$, hence by Theorem 5.10 g_k is integrable in the sense of Dunford and Schwartz. By Theorems 5.9 and 2.2 (ii) we have

$$\begin{aligned} \int_E g_k dv(x^*m) &= \int_E |f_k - f| dv(x^*m) \\ &\leq 4 \sup_{\substack{F \subseteq E \\ F \in \Sigma}} \left|\int_F (f_k - f) dx^*m\right| \\ &\leq 4 \sup_{\substack{F \subseteq E \\ F \in \Sigma}} \left|\int_F (f_k - f) dm\right|, \end{aligned}$$

if $x^* \in X_1^*$. Since f and (f_k) satisfy property B, it follows that given $\varepsilon > 0$, there is an n such that if $k \geq n$ and $x^* \in X_1^*$, then for every $E \in \Sigma$,

$$\int_E g_k dv(x^*m) < \varepsilon.$$

Since (h_{kj}) determines g_k, it follows in the same manner that for a fixed $k \geq n$, there is an ℓ such that $j \geq \ell$ implies that for every $E \in \Sigma$,

$$\int_E |h_{kj} - g_k| dv(x^*m) < \varepsilon,$$

provided $x^* \in X_1^*$. By omitting the first $\ell-1$ of the functions h_{kj} if necessary, we therefore conclude that for $x^* \in X_1^*$, $E \in \Sigma$, and $k \geq n$,

$$\int_E h_{kj}\,dv(x^*m) < 2\varepsilon .$$

Since h_{kj} is a nonnegative simple function the sets $E_{kj} = [h_{kj} > \gamma]$ belong to Σ for every $\gamma > 0$. Moreover,

$$\gamma v(x^*m, E_{kj}) \le \int_S h_{kj}\,dv(x^*m) < 2\varepsilon,$$

so

$$v(x^*m, E_{kj}) \le 2\varepsilon/\gamma,$$

if $x^* \in X_1^*$. By Theorem 2.2 (iii) we conclude that

$$\tilde{m}(E_{kj}) \le \sup_{X_1^*} v(x^*m, E_{kj}) \le 2\varepsilon/\gamma.$$

Since $m - \lim h_{kj} = g_k$, there is an $i \in N$ and a set $F_k \in \Sigma$ such that $\tilde{m}(F_k) < \varepsilon/\gamma$ and $|h_{ki}(s) - g_k(s)| \le \gamma$ if $s \notin F_k$. Therefore, if $s \notin E_{ki} \cup F_k$, we have

$$\begin{aligned} |f_k(s) - f(s)| &= g_k(s) \\ &\le |g_k(s) - h_{ki}(s)| + |h_{ki}(s)| \\ &\le 2\gamma , \end{aligned}$$

and $\tilde{m}(E_{ki} \cup F_k) \le 3\varepsilon/\gamma$. Now if δ_1 and $\delta_2 > 0$ are given, then choose $\gamma > 0$ such that $2\gamma < \delta_1$. Then choose $\varepsilon > 0$ so that $3\varepsilon/\gamma < \delta_2$. It follows from the arguments above that there is an $n \in N$ and sets G_k such that if $k \ge n$, then $\tilde{m}(G_k) < \delta_2$ and $|f_k(s) - f(s)| \le \delta_1$ if $s \notin G_k$. We conclude that $m - \lim f_k = f$.

To see that (ii) holds we note that the argument in the proof of (ii) implies (i) for Theorem 5.2 applies.

Conversely, if (i) and (ii) hold, then for each k there is a simple function g_k such that $\tilde{m}[|g_k - f_k| > 1/k] < 1/k$ and for every $E \in \Sigma$,

$$\left|\int_E (g_k - f_k)\,dm\right| < 1/k.$$

As in the proof of Theorem 4.6, it follows that $m - \lim g_k = f$. Suppose $\epsilon > 0$. Choose n such that $1/n < \epsilon$. Then choose $\delta > 0$ such that $\tilde{m}(E) < \delta$ implies

$$\left|\int_E g_k dm\right| < \epsilon \text{ and } \left|\int_E f_i dm\right| < \epsilon,$$

for $1 \le k \le n$ and $i \in N$. Then if $\tilde{m}(E) < \delta$,

$$\begin{aligned}\left|\int_E g_k dm\right| &\le \left|\int_E (g_k - f_k) dm\right| + \left|\int_E f_k dm\right| \\ &\le 2\epsilon\end{aligned}$$

for every k, so the indefinite integrals of the g_k's are uniformly m-continuous. It follows that $f \in L(m)$. Since

$$\begin{aligned}\left|\int_E f_k dm - \int_E f dm\right| &\le \left|\int_E (f_k - g_k) dm\right| + \left|\int_E (g_k - f) dm\right| \\ &\le 1/k + \left|\int_E (g_k - f) dm\right|,\end{aligned}$$

we conclude that f and (f_k) satisfy property B. □

We now state two important corollaries to Theorem 6.1 which are stronger forms of the results stated by Bartle [1].

6.2 Corollary. (Dominated Convergence) Suppose $(f_k) \subseteq L(m)$ and $g \in L(m)$. If $|f_k| \le |g|$ on S for every $k \in N$, then $m - \lim f_k = f$ if and only if $f \in L(m)$ and $f, (f_k)$ satisfy property B.

Proof. Suppose $m - \lim f_k = f$. By Lemma 5.11 the indefinite integrals of the f_k's are uniformly m-continuous, so the conclusion follows from Theorem 6.1. The converse is immediate by this same theorem. □

6.3 Corollary. (Bounded Convergence) Suppose $(f_k) \subseteq M$ is a sequence of functions uniformly bounded on S. Then $m - \lim f_k = f$ if and only if $f \in L(m)$ and $f, (f_k)$ satisfy property B.

Proof. Since constant functions are integrable, the conclusion follows from Corollary 6.2. □

7. L_p^o Spaces. As an application of the previous theory, we present a portion of L_p space theory. In this section Σ is an algebra and $m:\Sigma \to X$ is a bounded measure.

Recall the following set of inequalities, which were used in the proof of Lemma 5.11. If $f \in L(m)$ and $E \in \Sigma$, then

$$|\int_E f dm| = \sup_{X_1^*} |\int_E f dx^*m|$$

$$\leq \sup_{X_1^*} \int_E |f| dv(x^*m)$$

$$\leq 4 \sup_{F \subseteq E} \sup_{X_1^*} |\int_F f dx^*m|$$

$$= 4 \sup_{F \subseteq E} |\int_F f dm| .$$

These terms are finite by Corollary 5.7. Let $f \in M$. If $1 \leq p < \infty$ and $|f|^p \in L(m)$, define

$$\|f\|_p = \sup_{X_1^*} [\int_S |f|^p dv(x^*m)]^{1/p} .$$

Let L_p^o denote the space of all functions $f \in M$ such that $|f|^p \in L(m)$. Since $F \subseteq E$ implies that

$$|\int_F f dm| \leq \sup_{X_1^*} \int_E |f| dv(x^*m),$$

the inequalities above show that the two expressions

$$\sup_{E \in \Sigma} [\int_E |f|^p dm]^{1/p} \quad \text{and} \quad \sup_{X_1^*} [\int_S |f|^p dv(x^*m)]^{1/p}$$

are equivalent "norms" for the spaces L_p^o. We prefer to use the second of these expressions, since the properties of the scalar norms in $L_p(x^*m)$ are then available.

By Corollary 5.13, it follows that $L(m) = L_1^o$. Moreover, since $1 \le p \le q < \infty$ implies that $|f|^p \le 1 + |f|^q$, we see that $f \in L_q^o$ implies that $f \in L_p^o$, by Theorem 5.12. Hence $L_1^o \supseteq L_p^o \supseteq L_q^o$.

From the elementary inequality $|a+b|^p \le 2^p(|a|^p + |b|^p)$, it follows that $|f+g|^p \le 2^p(|f|^p + |g|^p)$; if f and g are in L_p^o, then so is f + g, by Theorem 5.12. L_p^o is clearly closed under scalar multiplication, and hence is a linear space. It is easy to see that $\|af\|_p = |a|\,\|f\|_p$ for any scalar a. Moreover,

$$\|f+g\|_p = \sup_{X_1^*}[\int_S |f+g|^p dv(x^*m)]^{1/p}$$

$$\le \sup_{X_1^*}[(\int_S |f|^p dv(x^*m))^{1/p} + (\int_S |g|^p dv(x^*m))^{1/p}]$$

$$\le \|f\|_p + \|g\|_p.$$

Since $\|f\|_p = 0$ if $f = 0$, we conclude that $\|\cdot\|_p$ is a semi-norm on L_p^o.

Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. The Hölder inequality shows that

$$\frac{|fg|}{\|f\|_p\|g\|_q} \le \frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q^q} ,$$

if $f \in L_p^o$ and $g \in L_q^o$. By Theorem 5.12 we conclude that $fg \in L_1^o$. If $x^* \in X_1^*$ we have

$$\int_S |fg|\,dv(x^*m) \le \|f\|_p \|g\|_q (1/p + 1/q).$$

Hence $\|fg\|_1 \le \|f\|_p \|g\|_q$.

A sequence of functions (f_k) in L_p^o is said to converge in L_p^o to a function $f \in L_p^o$ if we have $\lim \|f_k - f\|_p = 0$. The following convergence theorem is a consequence of Theorem 6.1.

7.1 Theorem. Suppose $1 \le p < \infty$ and (f_k) is a sequence in L_p^o. Then $f \in L_p^o$ and $\lim \|f_k - f\|_p = 0$ if and only if

(i) $m - \lim f_k = f$.

(ii) The indefinite integrals of the $|f_k|^p$'s are uniformly m-continuous.

Proof. Suppose $f \in L_p^o$ and $\lim \|f_k - f\|_p = 0$. If we set $g_k = |f_k - f|^p \in L(m)$, then the first part of the proof of Theorem 6.1 is easily adapted to show that $m - \lim f_k = f$. Since the indefinite integral of $|f|^p$ is m-continuous, and $\|f_k \chi_E\|_p \le \|f_k - f\|_p + \|f\chi_E\|_p$, (ii) follows as in the proof of Lemma 5.2.

Conversely, suppose that (i) and (ii) hold. By Theorem 4.6, $|f_k|^p$ converges to $|f|^p$ in measure, so by Theorem 6.1, $f \in L_p^o$. Moreover, Theorem 4.6 implies that $m - \lim |f_k - f|^p = 0$. Since $|f_k - f|^p \le 2^p(|f_k|^p + |f|^p)$, we have

$$\left|\int_E |f_k - f|^p dm\right| \le 2^{p+2} \sup_{\substack{F \subseteq E \\ F \in \Sigma}} \left|\int_F (|f_k|^p + |f|^p)\,dm\right|,$$

by Lemma 5.11. Since the indefinite integrals of the $|f_k|^p$'s are uniformly m-continuous, we conclude that the indefinite integrals of the $|f_k - f|^p$'s are also uniformly m-continuous.

By Theorem 6.1, then,

$$\lim \int_E |f_k - f|^p dm = 0$$

uniformly for $E \in \Sigma$. Finally, since

$$\|f_k - f\|_p^p \le 4 \sup_{E \in \Sigma} \left|\int_E |f_k - f|^p dm\right|,$$

we have $\lim \|f_k - f\|_p = 0$. □

The inequalities at the beginning of this section show that convergence in L_1^o is equivalent to property B. For if $f, (f_k)$ belong to $L_1^o = L(m)$, then

$$\sup_{E \in \Sigma} \left|\int_E (f_k - f)\, dm\right| \le \|f_k - f\|_1 \le 4 \sup_{E \in \Sigma} \left|\int_E (f_k - f)\, dm\right|.$$

Just as Corollary 6.2 followed from Theorem 6.1, we have:

7.2 <u>Corollary</u>. Suppose $1 \le p < \infty$, $(f_k) \subset L_p^o$, $g \in L_p^o$, and $|f_k| \le |g|$ on S for each k. Then $m-\lim f_k = f$ if and only if $f \in L_p^o$ and $\lim \|f_k - f\|_p = 0$.

<u>Proof</u>. Since $|f_k|^p \le |g|^p$, the indefinite integrals of the $|f_k|^p$'s are uniformly m-continuous. The conclusions follow from Theorem 7.1. □

7.3 <u>Corollary</u>. Suppose $1 \le p < \infty$. The subspace of simple functions is dense in L_p^o.

<u>Proof</u>. Suppose $f \in L_p^o$. Since $f \in M$ there is a sequence (f_k) of simple functions converging to f in measure, and by Theorem 4.7 we may assume that $|f_k| \le 2|f|$ on S for each k. By Corollary 7.2 it follows that $\lim \|f_k - f\| = 0$. □

8. Integration by Countably Additive Measures. Integration of scalar functions by a countably additive measure $m:\Sigma \to X$, where Σ is a σ-algebra, has been studied extensively by Bartle, Dunford and Schwartz [2], and also by Lewis [20]. The Bartle, Dunford and Schwartz Theory uses a result concerning the equivalence of weak compactness in the space of scalar measures with the existence of a control measure (as in Theorem 3.3), and the Vitali-Hahn-Saks Theorem. By means of these powerful results the following definition of the integral yields a theory which coincides with that in Section 5 for countably additive measures.

A sequence of functions $(f_k) \subseteq F(S)$ is said to converge (pointwise) almost everywhere to a function f if there is a null set A such that $\lim f_k(s) = f(s)$ for every $s \notin A$. In this case we write $ae - \lim f_k = f$.

8.1 Definition. $f \in F(S)$ is integrable if there is a sequence (f_k) of simple functions such that $ae - \lim f_k = f$, and such that for every $E \in \Sigma$, the sequence $(\int_E f_k dm)$ converges in X.

Pointwise almost everywhere convergence replaces convergence in measure in the countably additive situation by virtue of the nonnegative control measure $\underline{m}$. Since m and $\underline{m}$ are mutually continuous and countably additive, the standard theorems on the relation between convergence in measure and almost everywhere convergence can be shown to hold. In particular, $ae - \lim f_k = f$ implies $m - \lim f_k = f$. Moreover, since the indefinite integrals of the function f_k in Definition 8.1 are $\underline{m}$-continuous measures, the requirement that

$(\int_E f_k dm)$ converges for each $E \in \Sigma$, together with the Vitali-Hahn-Saks Theorem, show that the indefinite integrals of the f_k's are uniformly m-continuous, as in Definition 5.4.

8.2 <u>Remark</u>. If f is finitely additive, Σ is a σ-algebra, and m is s-bounded, so that there is a control measure $\underline{m}$ by Theorem 3.3, then we may apply Theorem 3.5. Thus if $(\int_E f_k dm)$ converges for every E, we conclude that the indefinite integrals of the f_k's are uniformly m-continuous. Convergence in measure must be retained however.

Bartle, Dunford and Schwartz establish a one-sided form of Theorem 6.1: If $(f_k) \subseteq L(m)$, $\text{ae}-\lim f_k = f$, and the indefinite integrals of the f_k's are uniformly m-continuous, then $f \in L(m)$ and $\lim \int_E f_k dm = \int_E f dm$ for $E \in \Sigma$. In view of Theorem 6.1 and the Vitali-Hahn-Saks Theorem we have the following extension:

8.3 <u>Theorem</u>. Suppose $(f_k) \subseteq L(m)$ and $\text{ae}-\lim f_k = f$. Then $f \in L(m)$ and $f, (f_k)$ satisfy property B if and only if $\lim \int_E f_k dm$ exists for each $E \in \Sigma$.

In the case of convergence in L_p^o we have by Theorem 7.1:

8.4 <u>Theorem</u>. Suppose $1 \le p < \infty$, $(f_k) \subseteq L_p^o$, and $\text{ae}-\lim f_k = f$. Then $f \in L_p^o$ and $\lim \|f_k - f\|_p = 0$ if and only if $\lim \int_E |f_k|^p dm$ exists for every $E \in \Sigma$.

For future use we state the following useful approximation theorem for the countably additive case.

8.5 Theorem. (Dunford and Schwartz [15]) Suppose $1 \le p < \infty$, Σ_o is an algebra, and $\Sigma = \sigma(\Sigma_o)$. Then the space of Σ_o-simple functions is dense in L_p^o.

Lewis [20] has developed an integration theory for countably additive measures with values in a locally convex space X using the Pettis approach of employing linear functionals. This integration theory coincides with that presented above when X is a Banach space.

9. A Riemann-type Integral. Gould [16] introduced an integral for scalar functions, which in the case of bounded functions is defined as a limit of Riemann-Stieltjes-type sums. Although he calls his integral an improper integral, it is in fact equivalent to the "proper" Bartle integral when m is a bounded measure. This alternative approach to integration has been extended by McShane [22] to obtain a very general theory. We identify the Gould and the Bartle integrals as the (G) and the (B) integrals respectively.

In this section Σ is an algebra and $m:\Sigma \to X$ is a bounded measure. If P and P′ are two partitions in $\pi(S)$, we write $P' \ge P$ to mean that each set in P′ is a subset of a set in P; we say that P′ is a refinement of P. If $P \in \pi(S)$ and $P = \{E_1,\dots,E_n\}$, then for a function $f \in F(S)$ we introduce the symbol $S(f,P)$ to indicate any Riemann-Stieltjes-type sum

$$\sum_{i=1}^{n} f(s_i)m(E_i),$$

where $s_i \in E_i$, $1 \le i \le n$. Let $B(S,\Sigma) = B(S)$ denote the family of all bounded measurable functions on S.

9.1 <u>Definition</u>. Suppose $f \in B(S)$. f is integrable (G) if for any $\varepsilon > 0$ there is a $P \in \pi(S)$ such that if $P' \in \pi(S)$ and $P' \geq P$, then $|S(f,P) - S(f,P')| < \varepsilon$.

We omit the proof of the following Theorem.

9.2 <u>Theorem</u>. Suppose $f \in B(S)$. f is integrable (G) if and only if there is an $x \in X$ with the following property: For every $\varepsilon > 0$ there is a $P \in \pi(S)$ such that if $P' \in \pi(S)$ and $P' \geq P$, then $|S(f,P') - x| < \varepsilon$.

The vector x is called the integral of f over S. We now show that for bounded measurable functions the (G) and (B) integrals coincide.

9.3 <u>Theorem</u>. If $f \in B(S)$, then f is integrable (G) and the (G) and (B) integrals coincide.

<u>Proof</u>. By Theorem 5.6, $f \in L(m)$; let $x = \int_S f dm$, and let (f_k) be a sequence of simple functions converging to f in measure. By Theorem 4.7 we may assume that $|f_k| \leq 2|f|$ on S for each k. Suppose that $|f| \leq K$ on S, and fix $\varepsilon > 0$. Choose $k \in N$ and $A \in \Sigma$ such that $\tilde{m}(A) < \varepsilon/4K$ and $|f(s) - f_k(s)| < \varepsilon/4\tilde{m}(S)$ if $s \notin A$. Suppose $f_k = \sum_{i=1}^{\ell} a_i \chi_{E_i}$. Let $F_1 = A$ and $F_i = E_{i-1} \setminus A$, $2 \leq i \leq \ell + 1$. Then $P = (F_i)$ is a partition of S and f_k is constant over $F_2, \ldots, F_{\ell+1}$. If $P' = (B_i) \geq P$ and if $f_k(s) = b_i$ for $s \in B_i$, then

$$\begin{aligned} |S(f,P') - \int_S f_k dm| &= |\sum_{i=1}^{n} f(s_i) m(B_i) - \sum_{i=1}^{n} \int_{B_i} f_k dm| \\ &\leq |\sum_{B_i \cap A = \emptyset} (f(s_i) - b_i) m(B_i)| + |\sum_{B_i \subset A} (f(s_i) - b_i) m(B_i)| \\ &\leq 4[\varepsilon/4\tilde{m}(S)]\tilde{m}(S \setminus A) + 4K\tilde{m}(A) + 8K\tilde{m}(A) \\ &\leq 4\varepsilon, \end{aligned}$$

where we have used Lemma 2.1.

Since $|f_k| \leq 2|f|$, Corollary 6.3 shows that f and (f_k) satisfy property B. There is then a $k \in N$ such that $|\int_S f_k dm - x| < \varepsilon$. The argument above shows that there is a partition $P \in \pi(S)$ such that if $P' \geq P$ then $|S(f,P') - x| \leq 5\varepsilon$. We conclude that f is integrable (G) and that the (G) and (B) integrals coincide. □.

The vector x will now be denoted by $\int_S fdm$. If $E \in \Sigma$ and $f \in B(S)$, then $f\chi_E$ is bounded and measurable by Theorem 4.6, and we write

$$\int_E fdm = \int_S f\chi_E dm .$$

Suppose $f \in M(m)$. The definition of measurability implies that f is bounded except perhaps on sets of arbitrarily small variation. Let B(f) denote the family of all sets in Σ on which m is bounded. It is easy to see that B(f) is a ring of sets. Since f is integrable on each set in B(f), the class $I(f) = \{\int_E fdm : E \in B(f)\}$ is nonvoid. Moreover, I(f) is a net in X over the directed set B(f), where $E \leq F$ means $E \subseteq F$. With this motivation we can define the integral for unbounded measurable functions.

9.4 Definition. $f \in M$ is integrable (G) if for every $\varepsilon > 0$ there is an $E \in B(f)$ such that if $F \in B(f)$ and $E \cap F = \emptyset$, then $|\int_F fdm| < \varepsilon$.

In view of the fact that I(f) is a net we have the following result due to Gould [16].

9.5 Theorem. (Gould) $f \in M$ is integrable (G) if and only if there exists an $x \in X$ having the following property: For every $\epsilon > 0$ there is an $E \in B(f)$ such that if $F \in B(f)$ and $E \subseteq F$, then $|\int_F f dm - x| < \epsilon$.

Proof. If such an x exists, then since $B(f)$ is a ring and the indefinite integral of f is a measure on $B(f)$, we have

$$|\int_F f dm| \leq |\int_{F+E} f dm - x| + |\int_E f dm - x|,$$

for every $E, F \in B(f)$ such that $E \cap F = \emptyset$. Fix $\epsilon > 0$ and choose $E \in B(f)$ such that if $G \in B(f)$ and $E \subseteq G$, then $|\int_G f dm - x| < \epsilon/2$. If $F \in B(f)$ and $E \cap F = \emptyset$, then $E \subseteq E + F \in B(f)$, and so

$$|\int_F f dm| < \epsilon.$$

Conversely, suppose f is integrable (G). Definition 9.4 implies that $I(f)$ is a Cauchy net in X. For suppose $\epsilon > 0$ is given. Choose $E \in B(f)$ such that if $F \in B(f)$ and $E \cap F = \emptyset$, then $|\int_F f dm| < \epsilon$. Now if $F, G \in B(f)$ and $F, G \geq E$, we have

$$\begin{aligned}|\int_F f dm - \int_G f dm| &\leq |\int_{F \setminus E} f dm| + |\int_{G \setminus E} f dm| \\ &\leq 2\epsilon,\end{aligned}$$

where we have again used the additivity of the integral. Let x denote the limit of $I(f)$. It is then immediate that x satisfies the requirement of the theorem. □

From this result we see that when f is integrable (G), then

$$\int_S f dm = \lim_{B(f)} \int_E f dm .$$

Suppose now that $f \in L(m)$. To see that f is integrable (G), suppose $\varepsilon > 0$, and choose $\delta > 0$ such that if $\tilde{m}(A) < \delta$, then $|\int_A f dm| < \varepsilon$. Since f is measurable, there is a set $E \in \Sigma$ such that f is bounded on E and $\tilde{m}(S \setminus E) < \delta$. It follows that for any $A \in \Sigma$ such that $A \cap E = \emptyset$, $\tilde{m}(A) < \delta$, and so $|\int_A f dm| < \varepsilon$. By Definition 9.4, f is integrable (G).

To see that integrability (G) implies integrability (B) we need the following result established by Gould [16].

9.6 Theorem. (Gould) If f is integrable (G), then the indefinite integral of f is a bounded m-continuous measure on Σ.

Proof. Since $I(f)$ is a Cauchy net it is a bounded set, so the indefinite integral is bounded. Additivity follows from the additivity of the indefinite integral for bounded functions. To see that the indefinite integral is m-continuous, suppose $\varepsilon > 0$. There is an $E \in B(f)$ such that if $F \in B(f)$ and $E \cap F = \emptyset$, then $|\int_F f dm| < \varepsilon$. Hence for $A \in \Sigma$, $A \cap E = \emptyset$ implies that $|\int_A f dm| \leq \varepsilon$. On E, however, the indefinite integral is already m-continuous by Theorem 5.6, so there is a $\delta > 0$ such that if $\tilde{m}(A) < \delta$ then $|\int_{A \cap E} f dm| < \varepsilon$. Thus if $\tilde{m}(A) < \delta$, then

$$|\int_A f dm| \leq |\int_{A \setminus E} f dm| + |\int_{A \cap E} f dm| \leq 2\varepsilon. \quad \square$$

We now prove a result that is the counterpart of Lemma 5.11.

9.7 Lemma. If $f, g \in M$, $|f| \le |g|$ on S, and g is integrable (G), then for every $A \in \Sigma$,

$$(*) \qquad \left|\int_A f dm\right| \le 4 \sup_{\substack{B \subseteq A \\ B \in \Sigma}} \left|\int_B g dm\right| .$$

Thus f is integrable (G).

Proof. Suppose $E \in B(f)$ and $\varepsilon > 0$. The indefinite integral of f over E is an m-continuous measure by Theorem 5.6, so there is a $\delta > 0$ such that if $\tilde{m}(A) < \delta$, then $\left|\int_{A \cap E} f dm\right| < \varepsilon$. Since g is bounded except on sets of arbitrarily small variation, there is a set $F \in B(g)$ such that $\tilde{m}(S \setminus F) < \delta$. Since f and g are both bounded on $E \cap F$, we have

$$\left|\int_{E \cap F} f dm\right| \le 4 \sup_{B \subseteq E \cap F} \left|\int_B g dm\right|$$

$$\le 4 \sup_{B \subseteq E} \left|\int_B g dm\right| ,$$

by Lemma 5.11. Therefore,

$$\left|\int_E f dm\right| \le \left|\int_{E \setminus F} f dm\right| + \left|\int_{E \cap F} f dm\right|$$

$$\le \varepsilon + 4 \sup_{B \subseteq E} \left|\int_B g dm\right| .$$

Since ε was arbitrary, we see that (*) holds for every $A \in B(f)$, and hence for every $A \in \Sigma$.

To see that (*) implies f is integrable (G), suppose $\varepsilon > 0$. Choose $E \in B(g)$ such that if $F \in B(g)$ and $E \cap F = \emptyset$, then $\left|\int_F g dm\right| < \varepsilon$. For any $B \in \Sigma$ with $B \cap E = \emptyset$, we have $\left|\int_B g dm\right| \le \varepsilon$. Since $B(g) \subseteq B(f)$, we conclude by (*) that if $A \in B(f)$ and $A \cap E = \emptyset$, then

$$\left|\int_A f dm\right| \le 4 \sup_{\substack{B \subseteq A \\ B \in \Sigma}} \left|\int_B g dm\right| \le 4\varepsilon . \ \square$$

Suppose now that f is integrable (G). Since $f \in M$, there is a sequence (f_k) of simple functions converging to f in measure, and by Theorem 4.7 we may assume that $|f_k| \le 2|f|$ on S. Since 2f is integrable (G), Lemma 9.7 shows that the indefinite integrals of the f_k's are uniformly m-continuous, since by Theorem 9.6 the integral of f is m-continuous. By Definition 5.4, $f \in L(m)$.

To see that the (G) and (B) integrals are equal, note that the definition of the (G) integral as the limit of I(f) shows that there is an increasing sequence of sets $E_k \in B(f)$ satisfying

$$(G) \int_S f dm = \lim \int_S f \chi_{E_k} dm.$$

Since the functions $f\chi_{E_k}$ are measurable by Theorem 4.6, and since we can choose the E_k's so that $\tilde{m}(S \setminus E_k) < 1/k$, it follows that $m - \lim f\chi_{E_k} = f$. Since $f \in L(m)$ and $|f\chi_{E_k}| \le |f|$ on S for each k, Corollary 6.2 implies that

$$(B) \int_S f dm = \lim \int_S f \chi_{E_k} dm.$$

The integrals therefore coincide.

CHAPTER II

Stieltjes Measures and Integrals

In this chapter we specialize the integration theory of Chapter I to the case where the measure is generated by a vector-valued function on an interval of the real line. We shall call such measures Stieltjes measures, and the resulting integrals will be called Stieltjes integrals. By restricting the Banach space in which the vector function takes its values to be an L_p space over a probability measure space, we shall study stochastic integration. (Chapter III). In the present chapter we shall investigate the question of boundedness of a Stieltjes measure, and the question of the existence of an extension to a countably additive measure on the Borel sets. Finally we discuss the important special case of Stieltjes measures with values in a Hilbert space.

1. Notation. Let T denote a closed, bounded interval in R, which for convenience we shall assume is the unit interval [0,1]. Let $\mathcal{S}$ denote the family of all subintervals of T of the form [a,b) or [a,1], together with $\emptyset$, where we shall always assume that $0 \le a \le b \le 1$. Finally, let Σ denote the family of all subsets of T which are of the form $A = \Sigma A_i$, where (A_i) is a finite disjoint family in $\mathcal{S}$. Note that Σ

is an algebra of subsets of T, and that the σ-algebra generated by Σ is the family of Borel sets in T, which is denoted by Σ_1.

Suppose that X is a Banach space, and that $z:T \to X$ is a function. Define a set function $m:\mathcal{S} \to X$ by setting $m[a,b) = z(b) - z(a)$ and $m[a,1] = z(1) - z(a)$. If $A \in \mathcal{S}$ and $A = \Sigma A_i$, where each $A_i \in \mathcal{S}$, then it follows at once by the definition of m on $\mathcal{S}$ that $m(A) = \Sigma m(A_i)$. Suppose now that $A \in \Sigma$, and that $\sum_{i=1}^{n} A_i$ and $\sum_{j=1}^{k} B_j$ are any two representations of A by finite sums of sets in $\mathcal{S}$. Then the sets $C_{ij} = A_i \cap B_j$ belong to $\mathcal{S}$ for each i and j, and we have

$$A_i = \sum_j C_{ij},\ 1 \le i \le n, \text{ and } B_j = \sum_i C_{ij},\ 1 \le j \le k.$$

By the preceding remark, it follows that

$$m(A_i) = \sum_j m(C_{ij}), \text{ and } m(B_j) = \sum_i m(C_{ij}),$$

for each i and j. We therefore conclude that

$$\sum_i m(A_i) = \sum_i \sum_j m(C_{ij}) = \sum_j m(B_j).$$

We have just shown that if $A \in \Sigma$ and $A = \Sigma A_i$ is any representation of A by a finite sum of sets in $\mathcal{S}$, then we can define $m(A) = \Sigma\, m(A_i)$, and this definition in independent of the representation of A. It is now immediate that m is finitely additive on Σ. In the sequel, we shall associate with each function $z:T \to X$ the unique measure m defined as above on Σ. When we wish to emphasize that the measure is generated by z, we write m_z. We call such a measure m the Stieltjes measure generated by z.

If m is a bounded Stieltjes measure on Σ, then m determines a class L(m) of functions integrable in the sense of Chapter I. The Σ-simple functions will now be called step functions; the class of step functions coincides with the class of all $\mathcal{S}$-simple functions. Every function that is the uniform limit of a sequence of step functions is obviously bounded and measurable with respect to any Stieltjes measure on Σ. Hence by Theorem I.5.6, such functions are integrable, and the sequence of integrals of the step functions converges to the integral of the limit function. If m has a countably additive extension to Σ_1, then Theorem I.8.5 shows that the class of step functions is still dense in L(m).

2. Bounded Stieltjes Measures. In this section $z:T \to X$ is a function generating the Stieltjes measure $m:\Sigma \to X$. We investigate conditions on z which insure that m is a bounded measure. By Remark I.2.3, m is bounded if and only if x^*m is bounded for each $x^* \in X^*$. Now since $x^*m_z[a,b) = x^*z(b) - x^*z(a) = m_{x^*z}[a,b)$, for every interval $[a,b)$ (or $[a,1]$) in $\mathcal{S}$, it follows that $x^*m_z = m_{x^*z}$ for every $x^* \in X^*$. By Theorem I.2.2 (ii) we know that the scalar measure m_{x^*z} is bounded if and only if it has bounded total variation. Recalling that each set in Σ is a finite sum of sets in $\mathcal{S}$, it follows that

$$\sup_{(A_i)\subseteq \mathcal{S}} \sum_{i=1}^{n} |m(A_i)| = \sup_{(A_i)\subseteq \Sigma} \sum_{i=1}^{n} |m(A_i)|,$$

where each (A_i) is a finite disjoint family. That is, the total variation of any measure m on Σ is the same as the

total variation of m restricted to $\mathcal{S}$. Therefore the measure m_{x^*z} is bounded if and only if

$$\sup_{(A_i)\subseteq \mathcal{S}} \sum_{i=1}^{n} |m_{x^*z}(A_i)| = \sup_{(A_i)\subseteq \mathcal{S}} \sum_{i=1}^{n} |x^*z(b_i) - x^*z(a_i)| < \infty,$$

where we write each A_i as $[a_i, b_i)$ (or $[a_i, b_i]$ if $b_i = 1$). But the right-hand term is just the total variation of the scalar function x^*z. Hence m_{x^*z} is bounded if and only if x^*z is a function of bounded variation for each $x^* \in X^*$. This establishes the following theorem.

2.1 Theorem. m_z is a bounded measure on Σ if and only if the function z is of weak bounded variation, in the sense that x^*z is a function of bounded variation for $x^* \in X^*$.

2.2 Remark. Since

$$\sup_{A\in\Sigma} |m(A)| = \sup_{(A_i)\subseteq \mathcal{S}} \left|\sum_{i=1}^{n} m(A_i)\right| = \sup \left|\sum_{i=1}^{n} z(b_i) - z(a_i)\right|,$$

where the supremum on the right is taken over all finite sets of points a_i and b_i, with $0 \le a_1 \le b_1 \le \ldots \le a_n \le b_n \le 1$, we have as a corollary to Theorem 2.1 that z is of weak bounded variation if and only if $\sup\left|\sum_{i=1}^{n} z(b_i) - z(a_i)\right| < \infty$. This result can also be proved directly using Lemma I.2.1 and the principle of uniform boundedness.

Let $C(T)$ denote the family of all scalar-valued continuous functions on T. If $f \in C(T)$, then since f is bounded and uniformly continuous, it follows from Theorems I.4.5 and I.5.6 that f is integrable with respect to any bounded measure on Σ. In fact, if m is a bounded measure and $f \in C(T)$, then in defining the integral according to Definition I.9.1

we may restrict the partitions P to be from $\pi(T,\mathcal{S})$. To see this we introduce the concept of the norm of a partition $P \in \pi(T,\mathcal{S})$. If $P = \{A_i : 1 \le i \le n\}$, where each A_i is an interval $[a_i, b_i)$ (and $A_n = [a_n, 1]$), then the norm of P, denoted by $|P|$, is defined as follows: $|P| = \max_{1\le i\le n} (b_i - a_i)$. We now state the above mentioned result.

2.3 Theorem. If m is a bounded measure and $f \in C(T)$, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that if P and P′ belong to $\pi(T,\mathcal{S})$ and $|P'|, |P| < \delta$, then $|S(f,P') - S(f,P)| \le \varepsilon$. In particular, this holds if $|P| < \delta$ and $P' \ge P$.

Proof. By the continuity of f, there is a $\delta > 0$ such that if $|s - t| < \delta$ and $s,t \in T$, then $|f(s) - f(t)| < \varepsilon/4\tilde{m}(T)$. Now if $P, P' \in \pi(T,\mathcal{S})$ and if $|P|, |P'| < \delta/2$, then

$$|S(f,P') - S(f,P)| = \left|\sum_{i=1}^{n} f(s_i)m(A_i) - \sum_{j=1}^{k} f(t_j)m(B_j)\right|$$

$$= \left|\sum_{i=1}^{n} f(s_i) \sum_{j=1}^{k} m(A_i \cap B_j) - \sum_{j=1}^{k} f(t_j) \sum_{i=1}^{n} m(A_i \cap B_j)\right|$$

$$= \left|\sum_{i=1}^{n} \sum_{j=1}^{k} (f(s_i) - f(t_j))m(A_i \cap B_j)\right|.$$

The only nonzero terms in this double sum occur when $m(A_i \cap B_j) \ne 0$; in this case $A_i \cap B_j \ne \emptyset$, and hence $|s_i - t_j| < \delta$. By Lemma I.2.1 we conclude that

$$|S(f,P') - S(f,P)| \le 4[\varepsilon/4\tilde{m}(T)]\tilde{m}(T) = \varepsilon.$$

The final statement of the theorem is immediate. □

Motivated by this result and the classical definition of the Riemann-Stieltjes integral, we make the following definition. Recall that $B(S)$ is the set of all bounded measurable functions on S.

2.4 Definition. Suppose $m:\Sigma \to X$ is a measure and $f \in B(S)$. Then $f \in RS(m)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $P, P' \in \pi(T,S)$ and $|P|, |P'| < \delta$, then $|S(f,P) - S(f,P')| < \varepsilon$.

Using this definition we could prove as in Theorem I.9.2 that if $f \in RS(m)$, then there is a vector $x \in X$ to which the sums $S(f,P)$ converge as $|P|$ converges to zero. When m is bounded, Theorem 2.2 shows that $C(T) \subseteq RS(m)$, and Theorem I.9.3 and Definition I.9.1 show that $RS(m) \subseteq L(m)$ and that the integrals coincide. In fact, if m is any measure from Σ to X, and if $C(T) \subseteq RS(m)$, then m is bounded. This follows from Remark I.2.3 and the following result.

2.5 Theorem. (Dunford [14]) If m is a scalar measure and $C(T) \subseteq RS(m)$, then m is bounded.

Proof. First we remark that if (P_n) is any sequence of partitions such that $|P_n|$ converges to zero, then for every $f \in C(T)$, the sums $S(f,P_n)$ converge to the integral of f with respect to m. Suppose that m is not bounded. Since m is a scalar measure we have only to suppose that the total variation of m is infinite. It follows that there is a

sequence (P_n) in $\pi(T,\mathcal{S})$ such that $|P_n| < 1/n$ and

$$\sum_{P_n} |m(A_i)| > n$$

for each n, where $P_n = (A_i)$; we may also assume that $m(A_i) \neq 0$. If for each n we choose a set of points $s_i \in A_i$, then by the preceding remark, $\lim S(f,P_n) = \int_T f dm$ for every $f \in C(T)$, where the sums $S(f,P_n)$ are formed using the points (s_i) chosen for each P_n.

Consider the mapping $U_n: C(T) \to X$ defined by $U_n(f) = S(f,P_n)$, $n \in N$. Each U_n is linear and bounded, and $\lim U_n(f)$ exists for each $f \in C(T)$. Since $C(T)$ with the norm $\|f\| = \sup_T |f(s)|$ is a Banach space, it follows by the principle of uniform boundedness that there is a constant $K > 0$ such that

$$(*) \qquad |S(f,P_n)| \le K\|f\|,$$

for every $f \in C(T)$. To obtain a contradiction we shall construct a function $f \in C(T)$ such that $\|f\| \le 1$, but for some $n \in N$, $|S(f,P_n)| > K$ holds. Choose n such that $\sum_{i=1}^{k} |m(A_i)| > K$, where $P_n = (A_i)$. Let s_i, $1 \le i \le k$, be the sequence chosen for P_n above. Define $f(s_i) = \overline{m(A_i)}/|m(A_i)|$, $1 \le i \le k$; $f(t) = f(s_1)$ for t in the interval $[0,s_1)$; $f(t) = f(s_k)$ for t in the interval $[s_k,1]$; and on each interval $[s_i,s_{i+1})$ define f to be linear between $f(s_i)$ and $f(s_{i+1})$. Then $f \in C(T)$, and we have $\|f\| = 1$; however,

$$S(f,P_n) = \sum_{i=1}^{k} f(s_i)m(A_i) = \sum_{i=1}^{k} |m(A_i)| > K.$$

Since this contradicts (*), m is bounded. □

The following theorem summarizes the results concerning the boundedness of Stieltjes measures.

2.6 Theorem. Let $m:\Sigma \to X$ be generated by z. The following statements are equivalent:

(i) z is of weak bounded variation.

(ii) m is bounded.

(iii) $C(T) \subseteq RS(m)$.

(iv) $C(T) \subseteq RS(x^*m)$ for every $x^* \in X^*$.

Proof. (i) $\Leftrightarrow$ (ii) follows from Theorem 2.1. Theorem 2.3 shows that (ii) $\Rightarrow$ (iii). We deduce from Theorems I.5.8 and 2.3 that (iii) $\Rightarrow$ (iv). Suppose that (iv) holds. Theorem 2.5 implies that x^*m is bounded; hence, by Remark I.2.3, m is bounded. $\square$

3. Extensions of Countably Additive Measures. If z is a scalar function on T, then the classical extension theorem (see Halmos [17]) for the measure m_z states that m_z has a unique countably additive extension to Σ_1, the Borel sets of T, if and only if z is left-continuous and of bounded (total) variation. When z is a vector function, we shall show that a similar theorem holds, provided that the Banach space X does not contain a copy of c_o. We need to introduce several preliminary concepts and results.

A semi-ring $\mathcal{S}$ is a family of subsets of S such that (i) $A,B \in \mathcal{S}$ implies that $A \cap B \in \mathcal{S}$; (ii) $A,B \in \mathcal{S}$ and $A \subseteq B$ implies that there is a finite family (C_i) in $\mathcal{S}$ such that

$A = C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n = B$ and $C_i \setminus C_{i-1} \in \mathcal{S}$ for $2 \leq i \leq n$ (see Halmos [17] or Dinculeanu [12] for information on semi-rings and measures on semi-rings). If $\mathcal{S}$ is a semi-ring, then the smallest ring $\mathcal{R}$ containing $\mathcal{S}$, which we call the ring generated by $\mathcal{S}$, is the family of all sums of finite disjoint subsets of $\mathcal{S}$. An example of a semi-ring is just the class $\mathcal{S}$ of half-open intervals discussed in Section 1. In this case the ring generated by $\mathcal{S}$ is the algebra Σ.

3.1 Remark. Suppose that $\mathcal{S}$ is a semi-ring and that $m:\mathcal{S} \to X$ is a set function. It is known (see Dinculeanu [12]) that if m is finitely additive on $\mathcal{S}$, then m has a unique finitely additive extension to $\mathcal{R}$, the ring generated by $\mathcal{S}$. Moreover, if m is countably additive on $\mathcal{S}$, then its extension to $\mathcal{R}$ is countably additive on $\mathcal{R}$ as well. From now on, given a finitely additive set function on $\mathcal{S}$, we shall assume that m has been extended to $\mathcal{R}$, and shall denote the extension by m. When we refer to the variation $\tilde{m}$ we always mean the variation of the extension, defined as in Chapter I, Section 2, by

$$\tilde{m}(A) = \sup_{\substack{B \subseteq A \\ B \in \mathcal{R}}} |m(B)|.$$

Note that if $\mathcal{R}$ is the ring generated by the semi-ring $\mathcal{S}$, then we also have

$$\tilde{m}(A) = \sup_{(B_i) \subseteq \mathcal{S}} \left| \sum_{i=1}^{n} m(B_i) \right|,$$

where (B_i) denotes a finite disjoint family in $\mathcal{S}$.

If S is a topological space and $A \subseteq S$, then we denote the interior of A by int A and the closure of A by cl A. If $\mathcal{S}$ is a semi-ring in S and $m:\mathcal{S} \to X$ is a finitely additive set function, we make the following definition, with Remark 3.1 in mind.

3.2 Definition. Suppose that S is a compact space. Then m is regular if for every $E \in \mathcal{S}$ and $\varepsilon > 0$, there are sets $A,B \in \mathcal{S}$ such that cl $A \subseteq E \subseteq$ int B and $\tilde{m}(B\setminus A) < \varepsilon$.

We now prove a result due to Huneycutt [18] which generalizes the classical Alexandroff Theorem for regular scalar measures (see Dunford and Schwartz [15]).

3.3 Theorem. (Huneycutt) If S is a compact space, $\mathcal{S}$ is a semi-ring in S, and $m:\mathcal{S} \to X$ is finitely additive and regular, then m is countably additive on $\mathcal{S}$.

Proof. Let (E_i) be a disjoint sequence in $\mathcal{S}$ such that $E = \Sigma E_i \in \mathcal{S}$, and fix $\varepsilon > 0$. By regularity there are sets $A,B \in \mathcal{S}$ with cl $A \subseteq E \subseteq$ int B and $\tilde{m}(B\setminus A) < \varepsilon$. There are also sequences of sets (A_i) and (B_i) in $\mathcal{S}$ such that for each i, cl $A_i \subseteq E_i \subseteq$ int B_i and $\tilde{m}(B_i\setminus A_i) < \varepsilon/2^i$. It follows that for each n,

$$\tilde{m}[(\bigcup_{i=1}^{n} B_i)\setminus A] < 2\varepsilon.$$

For

$$\begin{aligned}(\bigcup_{i=1}^{n} B_i)\setminus A &\subseteq [(\bigcup_{i=1}^{n} B_i)\setminus E] \cup [E\setminus A] \\ &\subseteq [\bigcup_{i=1}^{n} (B_i\setminus E_i)] \cup [E\setminus A],\end{aligned}$$

so

$$\tilde{m}[(\bigcup_{i=1}^{n} B_i)\setminus A] \le \tilde{m}[\bigcup_{i=1}^{n} (B_i\setminus E_i)] + \tilde{m}[E\setminus A]$$

$$\le \sum_{i=1}^{n} \varepsilon/2^i + \varepsilon < 2\varepsilon.$$

Since cl A is compact and cl $A \subset \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \operatorname{int} B_i$, there is a $k \in N$ such that $A \subset \bigcup_{i=1}^{k} B_i$. For $n \ge k$, therefore,

$$|m(A) - \sum_{i=1}^{n} m(E_i)| \le |m(\bigcup_{i=1}^{n} B_i) - m(\bigcup_{i=1}^{n} E_i)| +$$

$$+ |m(\bigcup_{i=1}^{n} B_i) - m(A)|$$

$$= |m[\bigcup_{i=1}^{n} B_i\setminus(\bigcup_{i=1}^{n} E_i)]| + |m[(\bigcup_{i=1}^{n} B_i)\setminus A]|$$

$$\le \tilde{m}[\bigcup_{i=1}^{n} (B_i\setminus E_i)] + \tilde{m}[(\bigcup_{i=1}^{n} B_i)\setminus A]$$

$$\le \sum_{i=1}^{n} \varepsilon/2^i + 2\varepsilon < 3\varepsilon.$$

Finally, $|m(E) - \sum_{i=1}^{n} m(E_i)| \le |m(E) - m(A)| +$

$$+ |m(A) - \sum_{i=1}^{n} m(E_i)|$$

$$\le \tilde{m}(E\setminus A) + 3\varepsilon \le 4\varepsilon,$$

if $n \ge k$. We conclude that $m(E) = \sum_{i=1}^{\infty} m(E_i)$; hence m is countably additive on $\mathcal{S}$. □

As an immediate corollary of this result we have:

3.4 <u>Corollary</u>. (Huneycutt [19]) If $m:\Sigma \to X$ is a finitely additive measure such that $\lim_{h\to 0^+} \tilde{m}[a-h,a) = 0$ for every $a \in (0,1]$, then m is countably additive on Σ.

Proof. It suffices to show that m is countably additive on $\mathcal{S}$, and in view of Theorem 3.3 we need only show that m is regular. Suppose that $[a,b) \in \mathcal{S}$. We may suppose that $0 < a < b < 1$, since the cases where $a = 0$ or $b = 1$ are proved in a similar fashion. Choose $h > 0$ such that $0 < a-h < a < b-h < b$. Then we have $[a,b-h) \subsetneq [a,b-h] \subsetneq [a,b) \subsetneq (a-h,b) \subsetneq [a-h,b)$, and $\tilde{m}[[a-h,b)\backslash[a,b-h)] \le \tilde{m}[a-h,a) + \tilde{m}[b-h,b)$. By hypothesis, given $\varepsilon > 0$, we can choose $h > 0$ such that $\tilde{m}[a-h,a) + \tilde{m}[b-h,b) < \varepsilon$. Therefore m is regular. $\square$

We now discuss an important class of Banach spaces characterized by Bessaga and Pelczynski [3]. The significance of this class for our purposes lies in the fact that it is precisely the class of range spaces for vector measures in which boundedness is equivalent to s-boundedness.

Let c_o denote the Banach space of all scalar sequences $x = (x_n)$ such that $\lim x_n = 0$, with norm defined by $|x| = \sup_n |x_n|$. If X is a Banach space and (x_n) is a sequence in X, we say that (x_n) is weakly Cauchy if the scalar sequences (x^*x_n) are Cauchy sequences for each $x^* \in X^*$. We say that (x_n) converges weakly to $x \in X$ if $\lim x^*x_n = x^*x$ for each x^*. Recall that a Banach space is said to be weakly sequentially complete if every weak Cauchy sequence converges weakly to some element in X. A series Σx_n in X is said to be weakly unordered bounded if for each $x^* \in X^*$ we have

$$\sup_{\Delta} \left| \sum_{i \in \Delta} x^*x_i \right| < \infty.$$

Note that by Lemma I.2.1, Σx_n is weakly unordered bounded if and only if $\Sigma|x^*x_i| < \infty$ for each $x^* \in X^*$.

If a Banach space X contains no subspace isometrically isomorphic to c_o, then we write $X \not\supseteq c_o$; otherwise we write $X \supseteq c_o$. We state the following important result without proof:

3.5 Theorem. (Bessaga and Pelczynski [3]) The following statements are equivalent:

(i) $X \not\supseteq c_o$.

(ii) Every weakly unordered bounded series in X converges unconditionally.

The following result was originally stated by Brooks and Walker [7] when X is weakly sequentially complete, and slightly extended to the present situation by Diestel [11].

3.6 Corollary. If $X \not\supseteq c_o$ and $m:\mathcal{R} \to X$ is any measure, then m is bounded if and only if m is s-bounded.

Proof. By Theorem I.3.2 an s-bounded measure is always bounded. Conversely, if m is bounded and (E_i) is a disjoint sequence in $\mathcal{R}$, then by Lemma I.2.1 we have

$$\begin{aligned}\sum_{i=1}^{n} |x^*m(E_i)| &\le 4|x^*| \sup_{\Delta\subseteq N_n} \Big|\sum_{i\in\Delta} m(E_i)\Big| \\ &\le 4|x^*| \sup_{E\in\mathcal{R}} |m(E)| < \infty,\end{aligned}$$

for every $n \in N$ and $x^* \in X^*$. It follows that the series

$\Sigma m(E_i)$ is weakly unordered bounded. By Theorem 3.5, therefore, $\Sigma m(E_i)$ converges unconditionally. By Theorem I.3.2 we conclude that m is s-bounded. □

3.7 Example. To see that Theorem 3.5 fails to hold when $X \supseteq c_o$, consider the sequence of unit vectors $e_n = (\delta_{ni})$ in c_o, where δ_{ni} is the Kronecker delta. The series Σe_n is weakly unordered bounded, since for any $\Delta \subset N$, $|\sum_{i\in\Delta} e_i| = 1$, but this series does not converge in c_o.

We now prove the final result that will be needed for the main theorem of this section.

3.8 Lemma. Suppose $m:\Sigma \to X$ is a bounded measure such that $\lim_{h\to 0^+} m[a-h,a) = 0$ for every $a \in (0,1]$. If $X \not\supseteq c_o$, then for every $a \in (0,1]$ we have

$$\lim_{h\to 0^+} \tilde{m}[a-h,a) = 0.$$

Proof. Since $\tilde{m}$ is monotone the limit above exists for each $a \in (0,1]$. Suppose there is an $a \in (0,1]$ and a $\delta > 0$ such that $\lim_{h\to 0^+} \tilde{m}[a-h,a) > \delta$. Let $n_1 = 1$. In view of Remark 3.1 there is an $n_2 > n_1$ and a finite disjoint family of sets $A_2,\ldots,A_{n_2}$ in $\mathcal{S}$, such that $|\sum_{i=n_1+1}^{n_2} m(A_i)| > \delta$. We may assume for convenience that $A_i = [a_i,b_i)$, where $a_2 < b_2 \le \ldots \le a_{n_2} < b_{n_2} \le a$. If $b_{n_2} = a$, then choose b such that $a_{n_2} < b < b_{n_2}$ and $|m[b,a)| < \delta/2$. This is possible by the continuity hypothesis on m. Then

$$\left|\sum_{i=n_1+1}^{n_2-1} m(A_i) + m[a_{n_2},b)\right| \ge \left|\sum_{i=n_1+1}^{n_2} m(A_i)\right| - |m[b,a)| > \delta/2.$$

Redefining $b_{n_2} = b$, so that $A_{n_2} = [a_{n_2}, b)$, we have

$$\left| \sum_{i=n_1+1}^{n_2} m(A_i) \right| > \delta/2.$$

Now since $\tilde{m}[b_{n_2}, a) > \delta$, we can repeat this argument over the interval $[b_{n_2}, a)$ to obtain an $n_3 > n_2$ and a finite disjoint family of sets $A_{n_2+1}, \ldots, A_{n_3}$ in $\mathcal{S}$, such that

$$\left| \sum_{i=n_2+1}^{n_3} m(A_i) \right| > \delta/2.$$

By induction, therefore, we obtain an increasing sequence (n_k) and a disjoint sequence (A_i) in $\mathcal{S}$ such that for $k \in N$,

$$\left| \sum_{i=n_k+1}^{n_{k+1}} m(A_i) \right| > \delta/2.$$

Define $E_k = \sum_{i=n_k+1}^{n_{k+1}} A_i$; then (E_k) is a disjoint sequence in Σ, and for every $k \in N$,

$$(*) \quad |m(E_k)| > \delta/2.$$

Since $X \not\supset c_o$ and m is bounded, m is s-bounded by Corollary 3.6. This contradicts $(*)$, so we conclude

$$\lim_{h \to 0^+} \tilde{m}[a-h, a) = 0. \ \square$$

The following theorem is our main result on the extension of Stieltjes measures to countably additive Borel measures.

3.9 Theorem. Suppose $X \not\supset c_o$, and let Σ_1 be the σ-algebra generated by Σ. The measure $m_z : \Sigma \to X$ has a unique countably additive extension to Σ_1 if and only if

(i) z is left continuous on $(0,1]$.

(ii) z is of weak bounded variation.

If $X \supseteq c_o$, then there is a function z satisfying (i) and (ii) such that m_z has no countably additive extension to Σ_1.

Proof. If m_z has a unique countably additive extension m_z' to Σ_1, then by Remark I.2.3, m_z' is bounded, so m_z is also bounded. By Theorem 2.1 z is of weak bounded variation. Since m_z is countably additive on Σ, $\lim m_z(A_i) = 0$ whenever (A_i) is a sequence in Σ such that $A_i \searrow \emptyset$. For any sequence $a_n \to b^-$, where $b \in (0,1]$, the sequence $[a_n, b)$ decreases to $\emptyset$. Hence we have $0 = \lim m_z[a_n,b) = \lim [z(b) - z(a_n)]$, so z is left continuous on $(0,1]$.

Conversely, if (i) and (ii) hold, then by Theorem 2.1, m_z is bounded. (i) implies $\lim_{h\to 0^+} m_z[a-h,a) = 0$ for every $a \in (0,1]$. By Lemma 3.8 we have $\lim_{h\to 0^+} \tilde{m}_z[a-h,a) = 0$ for $a \in (0,1]$. Consequently, by Corollary 3.4, m is countably additive on Σ. Since $X \not\supseteq c_o$ and m is bounded, m is s-bounded by Corollary 3.6. We conclude from Theorem I.3.4 that m_z has a unique countably additive extension to Σ_1.

Now suppose that $X \supseteq c_o$. Let (e_n) be the sequence of unit vectors in c_o defined in Example 3.7. Let

$$a_n = \begin{cases} e_1 & n = 1 \\ e_1 - \sum_{i=2}^{n} e_i & n \geq 2 \end{cases} .$$

Consider the function $z: T \to c_o$ defined by

$$z(t) = \sum_{n=1}^{\infty} a_n \chi_{(1/n+1,1/n]}(t)$$

for every $t \in T$. One can show that z is left continuous on $(0,1]$. Suppose that $0 \le s < t \le 1$. If $s = 0$, then since $z(0) = 0$, we have $z(t) - z(s) = a_n$, provided that $t \in (1/n+1, 1/n]$. If $s > 0$, then for $t \in (1/n+1, 1/n]$ and $s \in (1/n+k+1, 1/n+k]$, $z(t) - z(s) = a_n - a_{n+k} = \sum_{i=1}^{k} e_{n+i}$, provided $k \ge 1$. If $k = 0$ then $z(t) - z(s) = 0$. Now if $[a_1,b_1),\ldots,[a_k,b_k)$ are disjoint intervals in T, with $a_1 < b_1 \le \ldots \le a_k < b_k$, and if $a_1 > 0$, then there is a finite set $n_1 \le n_2 \le \ldots \le n_{2k}$ of integers such that

$$z(b_i) - z(a_i) = \sum_{j=n_{2i-1}}^{n_{2i}} e_j, \quad 1 \le i \le k.$$

If $a_1 = 0$, then $z(b_1) - z(a_1) = a_{n_2}$. In either case, since $\sum_{i=1}^{k} [z(b_i) - z(a_i)]$ is a finite sum of the form $\sum_{i=1}^{n} \varepsilon_i e_i$ for some n, where each ε_i is 0 or ± 1, we conclude that

$$\left|\sum_{i=1}^{k} [z(b_i) - z(a_i)]\right| \le 1.$$

By Remark 2.2, z is of weak bounded variation. Now if m_z has an extension to a countably additive measure on Σ_1, then m_z is s-bounded. Let $A_n = [1/n+1, 1/n)$ for each n. Then (A_n) is a disjoint sequence in $\mathcal{S}$, and $m_z(A_n) = z(1/n) - z(1/n+1) = e_n$, $n \in N$. Therefore $|m_z(A_n)| = 1$ for every n, so m_z is not s-bounded. Hence m_z cannot have a countably additive extension. □

The requirement that z be left continuous is in a sense superfluous; if $z(t^-)$ exists for every $t \in (0,1]$, then we can define $m[a,b) = z(b^-) - z(a^-)$, (or $m[0,b) = m(b^-) - m(0)$).

Essentially this is just normalizing the function z to be left continuous. Moreover, as the following result shows, if $z(t^-)$ exists at each point of $(0,1]$ and z is of weak bounded variation, then z determines a unique countably additive measure on Σ_1.

3.10 Theorem. If $z(t^-)$ exists for every $t \in (0,1]$ and if z is of weak bounded variation, then the function z' defined by $z'(t) = z(t^-)$, $t \in (0,1]$, and $z'(0) = z(0)$ is left continuous on $(0,1]$ and of weak bounded variation.

Proof. Since by definition z' is left continuous on $(0,1]$, it suffices to show that z' is of weak bounded variation. If $[a_1,b_1),\ldots,[a_n,b_n)$ are disjoint intervals in T with $a_1 < b_1 \le \ldots \le a_n < b_n$, then we can choose points $a_i' < a_i$ (if $a_1 = 0$ let $a_1' = 0$) and $b_i' < b_i$, such that for each i, $|z(a_i^-) - z(a_i')| < 1/2n$ and $|z(b_i^-) - z(b_i')| < 1/2n$. Then

$$\left|\sum_{i=1}^{n} z'(b_i) - z'(a_i)\right| \le \sum_{i=1}^{n} |z(b_i^-) - z(b_i')| + \sum_{i=1}^{n} |z(a_i^-) - z(a_i')| + \left|\sum_{i=1}^{n} z(b_i') - z(a_i')\right|$$

$$\le 1 + \tilde{m}_z(T).$$

By Remark 2.2 it follows that z' is of weak bounded variation. □

If $z:T \to X$ is a function such that m_z has a countably additive extension to Σ_1, then as we have already seen, z is left continuous. It follows that right-hand limits also exist at each point in $[0,1)$. For suppose $a \in [0,1)$ and $b_n \to a^+$. Then the intervals $[a,b_n)$ decrease to $\{a\} \in \Sigma_1$, and

so by the countable additivity of m_z, $m_z\{a\} = \lim m_z[a,b_n) = \lim[z(b_n) - z(a)]$. Thus $z(a^+)$ exists. By Theorems 3.10 and 3.9, every function z with right and left-hand limits that is of weak bounded variation determines a unique countably additive measure on Σ_1. Conversely, it is easy to see that if m is a countably additive measure on Σ_1 with values in X, then there is a function z with right and left-hand limits that is of weak bounded variation and satisfies $z(a^-) = m[0,a)$ if $a \in (0,1]$, and $z(a^+) = m[0,a]$ if $a \in [0,1)$. If we normalize this function to be left continuous and satisfy $z(0) = 0$, then we have established a one to one correspondence between the class of all countably additive measures on Σ_1 and the family of all normalized functions z that are of weak bounded variation. This is a generalization of the classical theorem relating countably additive Borel measures and functions of bounded variation in the scalar case.

4. Measures in Hilbert Space. In this section we apply our previous results to sharpen a theorem of Cramér [10], and discuss its consequences in integration theory. H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and $z:T \to H$ is a function which generates a measure $m_z = m:\Sigma \to H$.

Recall from Section 1 that the family $\mathcal{S}$ of all half-open intervals in T is a semi-ring which generates Σ. If $\mathcal{S}$ is any semi-ring, let $\mathcal{S}\times\mathcal{S}$ denote the class of all sets $A\times B$, where $A,B \in \mathcal{S}$. It is not difficult to see that $\mathcal{S}\times\mathcal{S}$ is

also a semi-ring. Elements of $\mathcal{S}\times\mathcal{S}$ are called rectangles. Consider the set function $m_1:\mathcal{S}\times\mathcal{S} \to \Phi$ defined by

$$m_1(A\times B) = \langle m(A), m(B)\rangle .$$

By the next lemma it will follow that m_1 is finitely additive on $\mathcal{S}\times\mathcal{S}$, and therefore by Remark 3.1, m_1 has a unique extension to a finitely additive measure on the algebra $\mathfrak{J}$ generated by $\mathcal{S}\times\mathcal{S}$. $\mathfrak{J}$ is just the family of finite sums of rectangles in $\mathcal{S}\times\mathcal{S}$.

4.1 Lemma. If $A\times B \in \mathcal{S}\times\mathcal{S}$ and $A\times B = A_1\times B_1 + A_2\times B_2$, then either $A = A_1 = A_2$ and $B = B_1 + B_2$, or $B = B_1 = B_2$ and $A = A_1 + A_2$.

Proof. If $A_1 = A_2$, then clearly $A = A_1 = A_2$. Suppose that $B_1 \cap B_2 \neq \emptyset$. If $b \in B_1 \cap B_2$, then for any $a \in A$, $(a,b) \in A_1\times B_1 \cap A_2\times B_2$, which is a contradiction. Hence $B_1 \cap B_2 = \emptyset$. Since $B \subseteq B_1 \cup B_2 \subseteq B$, $B = B_1 + B_2$.

If $A_1 \neq A_2$, then there are $a_1, a_2 \in A$ such that $a_1 \in A_1 \setminus A_2$ and $a_2 \in A_2 \setminus A_1$. Since (a_1,b_1) and $(a_2,b_2) \in A\times B$ for any $b_i \in B_i$, $i = 1,2$, (a_1,b_2) and $(a_2,b_1) \in A\times B$. Since $(a_1,b_2) \in A_1\times B_1$, $B_2 \subseteq B_1$. Similarly, $B_1 \subseteq B_2$, so $B = B_1 = B_2$. Then as above we must have $A_1 \cap A_2 = \emptyset$ and $A = A_1 + A_2$. □

Now suppose that $A\times B = A_1\times B_1 + A_2\times B_2$, and that $A = A_1 = A_2$ and $B = B_1 + B_2$. Then

$$\begin{aligned} m_1(A\times B) &= m_1(A\times(B_1 + B_2)) \\ &= \langle m(A), m(B_1 + B_2)\rangle \\ &= \langle m(A), m(B_1)\rangle + \langle m(A), m(B_2)\rangle \\ &= m_1(A_1\times B_1) + m_1(A_2\times B_2) . \end{aligned}$$

It follows that m_1 is finitely additive on $\mathcal{S}\times\mathcal{S}$, and so by Remark 3.1, we may assume that m_1 is a scalar measure on $\mathfrak{J}$.

4.2 Remark. Suppose that (A_i) is a finite disjoint family in $\mathcal{S}$. Then we have

$$\Big|\sum_{i=1}^{n} m(A_i)\Big|^2 = \Big\langle \sum_{i=1}^{n} m(A_i), \sum_{i=1}^{n} m(A_i)\Big\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle m(A_i), m(A_j)\rangle$$

$$\le \sum_{i=1}^{n}\sum_{j=1}^{n} |m_1(A_i \times A_j)| .$$

We conclude that if m_1 is a bounded measure on $\mathfrak{J}$, then m is a bounded measure on Σ, or equivalently z is of weak bounded variation.

Since m_1 is a scalar measure, Theorem I.2.2 (ii) implies that $\tilde{m}_1 \le v(m_1) \le 4\tilde{m}_1$ on $\mathfrak{J}$. It follows that in the definition of regularity, Definition 3.2, we may replace $\tilde{m}_1$ by $v(m_1)$. In what follows we shall use $v(m_1)$. The next lemma is the counterpart of Corollary 3.4.

4.3 Lemma. Suppose m_1 is bounded. Then m_1 is regular if

(i) $\lim_{s\to s_o^-} v(m_1,[s,s_o)\times[t,t_o)) = 0$, $s_o \in (0,1]$, $t,t_o \in T$.

(ii) $\lim_{t\to t_o^-} v(m_1,[s,s_o)\times[t,t_o)) = 0$, $t_o \in (0,1]$, $s,s_o \in T$.

Proof. Suppose $[a,b)\times[c,d) \in \mathcal{S}\times\mathcal{S}$, and $\varepsilon > 0$. By (i) there is a $\delta > 0$ such that $v(m_1,[a-\delta,a)\times[0,d)) < \varepsilon$ and $v(m_1,[b-\delta,b)\times[0,d)) < \varepsilon$. By (ii) there is an $\alpha > 0$ such

that $v(m_1,[0,b)\times[d-\alpha,d)) < \varepsilon$ and $v(m_1,[0,b)\times[c-\alpha,c)) < \varepsilon$. Now let $A = [a,b-\delta)\times[c,d-\alpha)$ and $B = [a-\delta,b)\times[c-\alpha,d)$. It follows that $\mathrm{cl}\, A \subseteq [a,b)\times[c,d) \subseteq \mathrm{int}\, B$, and that $B\backslash A \subseteq [a-\delta,a)\times[0,d) \cup [b-\delta,b)\times[0,d) \cup [0,b)\times[c-\alpha,c) \cup$ $\cup\ [0,b)\times[d-\alpha,d)$, so $v(m_1,B\backslash A) \leq 4\varepsilon$. We conclude that m_1 is regular, since the same type of argument can be applied to any set in $\mathcal{S}\times\mathcal{S}$. $\square$

We now apply Lemma 4.3 to obtain a connection between the countable additivity of m_1 and the continuity properties of the generating function z.

4.4 <u>Theorem</u>. Suppose that m_1 is bounded. Then $z(t^+)$ and $z(t^-)$ exist for every $t \in (0,1)$ (with one-sided limits at 0 and 1). m_1 is countably additive on $\mathfrak{J}$ if and only if z is left continuous.

<u>Proof</u>. Suppose that $\limsup_{s,t\to t_o^-} |z(t) - z(s)| > \delta > 0$. Then there is a sequence $s_1 < t_1 < s_2 < t_2 < \ldots < t_o$ such that $|z(t_i) - z(s_i)| \geq \delta$, $i \in N$. Since $|z(t_i) - z(s_i)|^2 = |m[s_i,t_i)|^2 = m_1[s_i,t_i)\times[s_i,t_i)$, and since the sequence $([s_i,t_i)\times[s_i,t_i))$ is disjoint, this contradicts the fact that m_1 is a bounded scalar measure, and hence has finite total variation. Therefore we have $\lim_{s,t\to t_o^-} |z(t) - z(s)| = 0$. Since H is complete there is then a $z(t_o^-) \in H$ such that $\lim_{t\to t_o^-} |z(t) - z(t_o^-)| = 0$. The case for limits from the right is handled similarly.

Suppose that m_1 is countably additive on $\mathfrak{J}$; then m_1 is continuous from above at $\emptyset$. Since $[t,t_o)\times[t,t_o)$ decreases to $\emptyset$ as $t \to t_o^-$, we conclude that

$$0 = \lim_{t\to t_o^-} m_1[t,t_o)\times[t,t_o)$$

$$= \lim_{t\to t_o^-} |z(t) - z(t_o)|^2 .$$

Therefore $z(t_o^-) = z(t_o)$.

Conversely, suppose that z is left continuous. In view of Lemma 4.3 and Theorem 3.3 we need only show

$$(*) \quad \lim_{s\to s_o^-} v(m_1,[s,s_o)\times[t,t_o)) = 0, \quad s_o \in (0,1], t,t_o \in T,$$

and similarly for $t \to t_o^-$ as in Lemma 4.3. By Schwartz's inequality we have

$$|m_1[s,s_o)\times[t,t_o)| = |\langle m[s,s_o),\ m[t,t_o)\rangle|$$

$$\leq |z(s_o) - z(s)||z(t_o) - z(t)|,$$

and so be the assumption of left continuity of z we see that

$$(**) \quad \lim_{s\to s_o^-} m_1[s,s_o)\times[t,t_o) = 0$$

for $s_o \in (0,1]$ and $t,t_o \in T$. Since $v(m_1)$ is monotone, the limit (*) exists for every $s_o \in (0,1]$ and $t,t_o \in T$. Suppose that for some choice of s_o, t, and t_o we have

$$\lim_{s\to s_o^-} v(m_1,[s,s_o)\times[t,t_o)) > \delta,$$

for some $\delta > 0$. Then there is a family of disjoint rectangles $[a_i,b_i)\times[c_i,d_i)$, $2 \leq i \leq n_2$, contained in $[0,s_o)\times[t,t_o)$, such that

$$\sum_{i=2}^{n_2} |m_1[a_i,b_i)\times[c_i,d_i)| > \delta.$$

We shall now show that we may assume $b_i < s_o$ for each i. Suppose that $[a,b)\times[c,d)$ is any rectangle in $[0,s_o)\times[t,t_o)$ such that $b = s_o$, and

$$|m_1[a,b)\times[c,d)| = \alpha > \varepsilon > 0,$$

for some $\varepsilon > 0$. By the continuity condition (**) we can select a b' such that $a < b' < s_o$ and

$$|m_1[b',s_o)\times[c,d)| < \alpha - \varepsilon.$$

Then $|m_1[a,b')\times[c,d)| \geq |m_1[a,b)\times[c,d)| - |m_1[b',s_o)\times[c,d)|$

$$> \alpha - (\alpha - \varepsilon) = \varepsilon.$$

Now if any of the $b_i = s_o$, then the argument above shows that we can find new b_i's such that $b_i < s_o$ and

$$\sum_{i=2}^{n_2} |m_1[a_i,b_i)\times[c_i,d_i)| > \delta.$$

Let $\delta_1 = \min\limits_{2\leq i\leq n_2} (s_o - b_i)$; then $v(m_1,[s_o-\delta_1,s_o)\times[t,t_o)) > \delta$, so we can repeat the process above. By induction we obtain a disjoint sequence $(A_i\times B_i)$ of rectangles and an increasing sequence (n_k) of integers (set $n_1 = 1$) such that

$$\sum_{i=n_k+1}^{n_{k+1}} |m_1(A_i\times B_i)| > \delta,$$

contradicting the boundedness of m_1. We conclude that (*) holds. The proof for $t\to t_o^-$ is similar to the above. □

The theorem we are about to prove was essentially stated by Cramér [10], but his statement and proof contain a gap which we now rectify. In our terminology, the statement of Cramér's result is as follows. Let $z:T \to H$ be a function generating m and m_1 as above. If m_1 is a bounded measure,

then m and m_1 have countably additive extensions to Σ_1 and $\mathfrak{J}_1$ respectively, where $\mathfrak{J}_1$ denotes the σ-algebra generated by $\mathfrak{J}$. Since even for scalar functions the measure m_z is not countably additive unless z is left continuous, the theorem as stated is not correct. We now present the corrected version.

4.5 Theorem. If m_1 is bounded, then the measures m and m_1 have unique countably additive extensions to Σ_1 and $\mathfrak{J}_1$ respectively if and only if z is left continuous. In this case we have

$$(*) \quad \langle m(A), m(B)\rangle = m_1(A \times B)$$

for every A and B in Σ_1, where we identify the extensions by m and m_1.

Proof. If m_1 has a countably additive extension then z is left continuous by Theorem 4.4. Conversely, if z is left continuous, then by this same theorem m_1 is countably additive on $\mathfrak{J}$. By the assumption of boundedness, m_1 has a unique extension to $\mathfrak{J}_1$ (by the classical extension theorem or Theorem I.3.4), and z is of weak bounded variation by Remark 4.2. By Theorem 3.9, m has a unique countably additive extension to Σ_1.

To see that (*) holds, fix $B \in \Sigma$, and define set functions μ and λ on Σ_1 as follows:

$$\mu(A) = m_1(A \times B), \; \lambda(A) = \langle m(A), m(B)\rangle,$$

for $A \in \Sigma_1$. Since m and m_1 are countably additive, it follows that μ and λ are finitely additive and continuous from above at $\emptyset$; thus μ and λ are countably additive scalar measures.

Since $\lambda(A) = \mu(A)$ for $A \in \Sigma$, it follows by the uniqueness of extensions that $\lambda(A) = \mu(A)$ for $A \in \Sigma_1$, or $m_1(A\times B) = \langle m(A), m(B)\rangle$ for $A \in \Sigma_1$ and $B \in \Sigma$ fixed and $B \in \Sigma_1$ fixed. Repeating this argument for $A \in \Sigma_1$ fixed and $B \in \Sigma_1$ gives the desired result. $\square$

The author is indebted to Professor Brooks for discussions on the previous material; in particular the proof of (*) in Theorem 4.5 is due to him.

As an application of Theorem 4.5, we discuss the most important case, namely functions z with orthogonal increments. A function $z: T \to H$ has orthogonal increments if $\langle z(t_1) - z(s_1), z(t_2) - z(s_2)\rangle = 0$ whenever $s_1 < t_1 \le s_2 < t_2$. The differences $z(t) - z(s)$ are called the increments of the function z. This property simply states that for disjoint sets $A, B \in \mathcal{S}$ we have $m_1(A\times B) = \langle m(A), m(B)\rangle = 0$; m_1 therefore reduces to a nonnegative measure on Σ if we define $m_1(A) = \langle m(A), m(A)\rangle = |m(A)|^2$. It follows from orthogonality that m_1 is finitely additive, since

$$\begin{aligned} m_1(A+B) &= \langle m(A+B), m(A+B)\rangle \\ &= \langle m(A), m(A)\rangle + \langle m(A), m(B)\rangle + \\ &\quad + \langle m(B), m(A)\rangle + \langle m(B), m(B)\rangle \\ &= m_1(A) + m_1(B). \end{aligned}$$

Since m_1 is nonnegative and finitely additive it is monotone, and so $m_1(A) \le m_1(T)$ for every $A \in \Sigma$. By Theorem 4.5 we conclude that m and m_1 have unique extensions to Σ_1 if and only if z is left continuous. Moreover, since m_1 is bounded, Theorem 4.4 implies that $z(t^+)$ and $z(t^-)$ $(z(0^+)$ and $z(1^-))$ always exist. It follows that any function z with orthogonal

increments determines countably additive measures m and m_1 if we normalize z to be left continuous. In fact, z is continuous except possibly at countably many points, since $F(t) = m_1[0,t)$ is a nondecreasing function and $|z(t) - z(s)|^2 = F(t) - F(s)$.

Suppose now that z is any function with orthogonal increments, not necessarily left continuous, and z generates the measures $m:\Sigma \to H$ and $m:\Sigma \to [0,\infty)$ as above. Since

$$[\tilde{m}(A)]^2 = \sup_{\substack{B\subseteq A\\ B\in\Sigma}} |m(B)|^2 = \sup_{\substack{B\subseteq A\\ B\in\Sigma}} m_1(B) = m_1(A),$$

it follows that convergence in measure with respect to m is equivalent to convergence in measure with respect to m_1. Moreover, if (f_k) is a sequence of step functions on T, then the indefinite integrals of the f_k's are uniformly m-continuous if and only if they are uniformly m_1-continuous. Suppose $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ is a step function; then

$$\left|\int_E f dm\right|^2 = \left\langle \sum_{i=1}^{n} a_i m(E \cap E_i), \sum_{i=1}^{n} a_i m(E \cap E_i)\right\rangle$$

$$= \sum_{i=1}^{n} |a_i|^2 m_1(E \cap E_i) = \int_E |f|^2 dm_1,$$

using the orthogonality of the measure m. By the remarks above, and by Definition I.5.4 and Theorem I.7.1, we conclude that $f \in L(m)$ if and only if $f \in L_2^o(m_1)$. Moreover, for step functions f and g it can be shown that

$$\left\langle \int_E f dm, \int_F g dm\right\rangle = \int_{E\cap F} f\bar{g} dm_1,$$

and so by continuity this equality holds for every f and g in

$L(m) = L_2^o(m_1)$. If z is left continuous, so that m and m_1 have countably additive extensions to Borel measures, then by Theorem I.8.5 the class of step functions is dense in $L(m) = L_2^o(m_1)$, and the results above still hold by continuity.

CHAPTER III

Stochastic Integration

In this chapter we shall restrict the integration theory of the preceding chapters to the case where the Banach space X is an L_p space over a probability measure space. Stieltjes integrals with respect to a function $z:T \to L_p$ are stochastic integrals in the sense that the integral is an element of L_p, or in probabilistic terms, a random variable. We show that the general integration theory of Chapters I and II includes certain stochastic integrals previously defined by other methods.

In Section 1 we introduce and discuss the probabilistic concepts which are used. Section 2 consists mainly of background material on the theory of stochastic processes. In Section 3 we attempt to motivate the study of stochastic integration, and then discuss the general integration theory in the present context. The sample path stochastic integral is studied in Section 4; we compute some statistical properties of the integral in a simple special case, and discuss an integration by parts formula. The Wiener-Doob stochastic integral in L_2 is discussed briefly in Section 5. In Section 6 we deal with martingale stochastic integrals, and prove a general existence theorem.

1. Probability Concepts and Notation. We say that (Ω,F) is a measurable space if Ω is a nonvoid set (ω is a generic element in in Ω) and F is a σ-algebra of subsets of Ω. If there is a countably additive measure P defined on F with values in $[0,1]$, and if $P(\Omega) = 1$, then we say that (Ω,F,P) is a probability measure space, and P is a probability measure. A probability measure space, or probability space, is therefore just a finite nonnegative measure space with the property that Ω has measure 1.

Suppose that (Ω,F) and (Ω',F') are two measurable spaces, and $x:\Omega \to \Omega'$ is a function. x is said to be measurable (relative to F and F') if the set $[x \in A'] \equiv x^{-1}(A')$ belongs to F for every $A' \in F'$. The family of all sets $[x \in A']$, $A' \in F'$, is a σ-algebra contained in F, and we denote it by $F(x)$. Note that $F(x)$ is the smallest σ-algebra F'' contained in F such that x is measurable relative to F'' and F'. If P is a measure on F, then a measurable function x determines a measure P' on F' by the relationship $P'(A') = P[x \in A']$ for each $A' \in F'$. If P is a probability measure on F then P' is a probability measure on F' as well. In this case P' is called the distribution of x. When $\Omega' = \Phi$ we shall always take F' to be the σ-algebra of Borel sets in Φ.

A measurable scalar-valued function on a probability space (Ω,F,P) is called a random variable. As motivation for this term, suppose we have a system whose state can be described by a scalar variable x, and whose behavior is subject to some sort of statistical variation. We imagine

the numbers x corresponding to the (random) states of the system to be the values of a function x defined on some probability space. The assumption that x is a measurable function insures that the probability of each event $x \in B$ is defined, where B is any Borel set. This probability is just $P[x \in B]$.

$L_p \equiv L_p(\Omega, F, P)$, where $1 \le p < \infty$, will denote the Banach space of all equivalence classes of random variables x such that

$$E|x|^p \equiv \int_\Omega |x(\omega)|^p P(d\omega) < \infty.$$

L_∞ denotes the space of equivalence classes of essentially bounded random variables. It is known (see Dunford and Schwartz [15]) that for $1 \le p < \infty$, L_p is weakly sequentially complete. Since c_o is not weakly sequentially complete, it follows that $L_p \not\supset c_o$; hence Theorem II.3.9 applies to these spaces. The number $E|x|^p$ is called the p-th absolute moment of x. If $k \in N$, then Ex^k is called the k-th moment of x. In particular, the first moment Ex is called the mean or expectation of x. It is well known from the theory of integration that the expectation operator E is a continuous linear functional on L_1. If $x \in L_2$, then the number

$$\sigma^2(x) = E|x - Ex|^2 = E|x|^2 - |Ex|^2$$

is called the variance of x, and represents in some sense the dispersion of x about its mean value Ex.

Standard references for results in probability theory are Loève [21] and Chung [9]; most of the material in this first section can be found in either of these works.

2. Stochastic Processes. In this section we discuss the definition of a stochastic process (random function) as a mapping from a subset T of R into an L_p space over a probability space. In order to illustrate some of the important properties that a process may have, we discuss two classical examples, the Poisson process and the Wiener process.

Let T be a subset of R whose elements are interpreted as points in time, and suppose we have a system whose observable state at time $t \in T$ is some scalar quantity $x(t)$. If the system is subject to influences of a random or statistical nature, then we are led to suppose that $x(t)$ is a random variable for each $t \in T$. Our mathematical model of the system is then a family of random variables $x(t)$, $t \in T$, which are defined on some underlying probability space (Ω, F, P). If we assume that each $x(t)$ belongs to L_p, so that the convergence properties of this space are available, then we have defined a mapping x from T into L_p. x is called a stochastic process or random function since it represents the behavior of a system subject to stochastic (random) influences.

Since L_p is a space of (equivalence classes of) functions from Ω into Φ, we may also think of a random function x as a mapping from $T \times \Omega \to \Phi$. Then each $\omega \in \Omega$ determines a function $x(\cdot, \omega): T \to \Phi$; this function is called the sample function or sample path corresponding to ω. Every set $A \in F$ determines a set of sample functions $\{x(\cdot, \omega): \omega \in A\}$, and we say that the probability of this set of sample functions is p if $P(A) = p$.

A fundamental problem in the theory of random functions concerns the fact that we would like to discuss the probabilities of sets of sample functions which may not correspond to sets in F as above. For example, given any finite set of times $t_1,\ldots,t_n$ and Borel sets $B_1,\ldots,B_n$, the set

$$[x(t_i) \in B_i : 1 \le i \le n] = \bigcap_{i=1}^{n} [x(t_i) \in B_i]$$

is in F. Even for countably many t_i's and B_i's such sets are in F. However, the set

$$[x(t) \in B : t \in (a,b)] = \bigcap_{t \in (a,b)} [x(t) \in B]$$

clearly need not be measurable. Doob [13] discovered a reasonable way to skirt this issue. He was able to show that every random function x is equivalent to a random function y which behaves nicely in this regard, where equivalence means that for every $t \in T$, $P[x(t) = y(t)] = 1$. We shall always assume for the sake of simplicity that we are dealing with such well-behaved processes.

In order to discuss two important examples of stochastic processes we need to introduce a few additional concepts. Suppose that $x(t)$, $t \in T$, is a random function, and $t_1,\ldots,t_n$ is a finite subset of T. Assume for simplicity that $x(t)$ is real-valued for each t. Let R^n denote real Euclidean n-space, and let B^n denote the σ-algebra of Borel sets in R^n. It can be shown that the family of all product sets $B_1 \times \ldots \times B_n$, where $B_i \in B^1$ for $1 \le i \le n$, is a semi-ring which generates B^n. Consider the mapping $\omega \to [x(t_1,\omega),\ldots,x(t_n,\omega)]$, which is a

function from Ω into R^n. Since each $x(t_i)$ is measurable, it follows that the set

$$[[x(t_1),\ldots,x(t_n)] \in B_1 \times \ldots \times B_n] = \bigcap_{i=1}^{n} [x(t_i) \in B_i]$$

belongs to F, for every product set $B_1 \times \ldots \times B_n$. As in Section 1, the mapping $\omega \to [x(t_1,\omega),\ldots,x(t_n,\omega)]$ determines a measure on the semi-ring of product sets, and by a standard extension theorem (see Halmos [17]), this measure extends to a probability measure $P_{t_1,\ldots,t_n}$ on B^n. The measure $P_{t_1,\ldots,t_n}$ is called the joint distribution of the random variables $x(t_1),\ldots,x(t_n)$. It seems reasonable, and in fact can be verified (see Loève [21]), that a random function is characterized by the family of all its joint distributions $P_{t_1,\ldots,t_n}$.

Suppose that $x(t)$, $t \in T$, is a random function. As in Section II.4, we say that $x(t_2) - x(t_1)$ is an increment of the process. $x(t)$ is said to have stationary increments if for every choice of $t > 0$ and $s_1 < t_1 \le s_2 < t_2 \le \ldots \le s_n < t_n$ in T, the joint distribution of the increments $x(t_i + t) - x(s_i + t)$, $1 \le i \le n$, coincides with the joint distribution of the increments $x(t_i) - x(s_i)$, $1 \le i \le n$. In other words, the probabilistic properties of the increments are invariant under a shift in time.

Recall that every random variable $x(t) - x(s)$ determines a σ-algebra $F(x(t) - x(s))$ as in Section 1. We say that $x(t)$, $t \in T$, has independent increments if for each choice of $s_1 < t_1 \le \ldots \le s_n < t_n$ in T and $A_i \in F(x(t_i) - x(s_i))$, $1 \le i \le n$, $P[\bigcap_{i=1}^{n} A_i] = P(A_1) \ldots P(A_n)$. In this case it is

known (see Loève [21]) that if $E \prod_{i=1}^{n} |x(t_i) - x(s_i)| < \infty$, then we have

$$E \prod_{i=1}^{n} [x(t_i) - x(s_i)] = \prod_{i=1}^{n} E[x(t_i) - x(s_i)].$$

Now we consider a classical random function called the Poisson process. Suppose we have a general situation where events occur in a random fashion that is in some sense uniform over $T = [0,1]$. Let $x(t)$ denote the number of events which have occurred in the interval $[0,t)$. It seems reasonable to assume that the random function x has the following properties:

(i) Independent increments. The increment $x(t) - x(s)$, $t > s$, represents the number of events occurring in $[s,t)$, and we would expect that over disjoint intervals of time, the numbers of events occurring in those intervals are independent of each other.

(ii) Stationary increments. The uniformity assumption means that over disjoint intervals of the same length, we expect that the distribution of occurrences is the same.

(iii) $x(0) \equiv 0$.

If in addition we assume that there is a $\lambda > 0$ such that for small values of $h > 0$ we have

$$P[x(h) = 1] = \lambda h + o(h),$$

and

$$P[x(h) = 0] = 1 - \lambda h + o(h),$$

then

$$P[x(h) > 1] = o(h),$$

and it can be shown (see Doob [13]) that for every $s < t$ in T, we have

$$P[x(t) - x(s) = n] = e^{-\lambda(t-s)} \cdot \frac{\lambda^n (t-s)^n}{n!},$$

for $n = 0,1,2,\ldots$. This discrete probability distribution is called the Poisson distribution, and the relation above states that the random variable $x(t) - x(s)$ is distributed according to a Poisson distribution, with parameter λ. The mean and variance of the increments are given by

$$E[x(t) - x(s)] = \lambda(t-s),$$

and

$$\sigma^2(x(t) - x(s)) = E[x(t) - x(s)]^2 - [E(x(t) - x(s))]^2 =$$
$$= \lambda(t-s).$$

The sample functions of a Poisson random function (except for a set of sample functions of measure zero) are monotone, nondecreasing, nonnegative, integer-valued functions of t, with at most finitely many jumps of unit magnitude in T. This might be expected in view of the event-counting interpretation of $x(t)$.

A second important example of a random function is the Wiener process, which has been used extensively as a mathematical model for Brownian motion. Suppose we observe a particle undergoing Brownian motion, and restrict our attention to, say, the x coordinate of its position. Due to the random bombardment of the particle by the fluid's molecules, this x coordinate changes in an erratic fashion as time progresses. If we denote the value of x at time $t \geq 0$ by

$x(t)$, then it seems reasonable to assume that $x(t)$ is a random function which has the following properties:

(i) Independent increments. The increment $x(t) - x(s)$ represents a change in position along the x-axis over the interval $[s,t)$. Over disjoint intervals of time these changes should be independent of each other due to the random character of the molecular bombardments.

(ii) Stationary increments. Since we would assume that the fluid in which the particle is suspended is homogeneous, the distributions of change over two intervals of the same length should be the same.

(iii) $x(0) \equiv 0$, assuming we use the position of our initial observation as the center of the coordinate system.

(iv) Normally (Gaussian) distributed increments, with mean zero and variance proportional to the length of the time interval. This follows from the assumption that the motion is random and continuous, and means that if $a < b$, then

$$P[a \le x(t) - x(s) < b] = \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \int_a^b \exp[-\tau^2/2\sigma^2(t-s)]d\tau,$$

where $s < t$, and $\sigma^2 > 0$ is some constant, namely the variance over an interval of unit length (we have assumed $t \ge 0$ so $T = [0,\infty)$).

It can be shown that except for a set of sample functions of measure 0, the sample functions of a Wiener process are continuous, nowhere differentiable, and of infinite variation over any finite interval. Although this last property runs

counter to what intuition says the sample paths should look like, the Wiener process has played a very significant role in the application of probability theory to physics and engineering. In fact, the first stochastic integrals were defined by Wiener for scalar functions with respect to a Wiener process.

A standard reference for results on stochastic processes is Doob [13]. Much of the material in this section can be found in his classic work.

3. Stochastic Integrals. We now discuss as example which motivates the theory of stochastic integration. Suppose we have a system consisting of a signal processing unit with an input signal $z(t)$ and an output signal $x(t)$. Suppose that the effect of the processor can be described by a scalar function of two variables $g(t,\tau)$ in the following way. A small change $z(d\tau) = z(\tau + d\tau) - z(\tau)$ in the input at time τ produces a change in the output of the system at a time $t \geq \tau$ given by $g(t,\tau)z(d\tau)$. Moreover, suppose that the response of the system is linear, in the sense that small disturbances $z(d\tau_1),\ldots,z(d\tau_n)$ produce a change in the output given by $\sum_{i=1}^{n} g(t,\tau_i)z(d\tau_i)$, for $t \geq \max \tau_i$. If we assume that $g(t,\tau) = 0$ if $t < \tau$, then for a given input function $z(t)$, we have formally that

$$x(t) = \int_{-\infty}^{\infty} g(t,\tau)z(d\tau).$$

As long as the functions g and z are sufficiently well-behaved scalar functions, the definition of this integral poses no problem.

Suppose, however, that the input $z(t)$ is a random function, for example, a signal contaminated by random noise. Then the output will also be random, and we are faced with the problem of how to define the integral above so that it is a random variable for each t. If the function g is continuous in τ for every t, and if almost every sample function of z is a function of bounded variation, then we could define the random output in a pointwise fashion as follows:

$$(*) \quad x(t,\omega) = \int_{-\infty}^{\infty} g(t,\tau)z(d\tau,\omega).$$

Stochastic integrals defined in this way are called sample path integrals; although they are the most restrictive kind of integral, their study does provide motivation for the more powerful kinds of stochastic integrals. In general, this simple solution to the problem is not feasible, since even processes as well-behaved as the Wiener process have almost every sample function of unbounded variation.

Using the integration theory of Chapters I and II when the Banach space X is an L_p space over a probability space, we obtain a very general definition of the stochastic integral. Suppose that $z:T \to L_p$ is a random function, where $1 \leq p < \infty$. As in Chapter II we can define a measure $m:\Sigma \to L_p$ by setting $m[a,b) = z(b) - z(a)$ on $\mathcal{S}$. For each $A \in \Sigma$, $m(A)$ is thus a random variable in L_p. The measure m is bounded if and only if z is of weak bounded variation, and by the form of $L_p{}^*$ this is equivalent, using Theorem II.2.1, to

requiring that the scalar functions $t \to E[z(t)\overline{x}]$ are of bounded variation for every $x \in L_p$, where $1/p + 1/q = 1$ if $1 < p < \infty$, and $q = \infty$ if $p = 1$.

Suppose that z is of weak bounded variation. Then z determines a class $L(z) \equiv L(m)$ of integrable scalar functions, and for $f \in L(z)$ we write

$$\int_T f(t)z(dt) = \int_T f(t)m(dt).$$

We shall call such integrals stochastic Stieltjes integrals, or norm integrals for short, since they are defined as the limit of a sequence of integrals of step functions in the L_p norm. If $\|x\|_p$ denotes the L_p norm $(E|x|^p)^{1/p}$ of an element $x \in L_p$, then it is known (see Loève [21]) that $1 \le r \le p$ and $x \in L_p$ implies that $\|x\|_r \le \|x\|_p$. This shows that $L_r \supseteq L_p$, and that the canonical embedding U_{pr} of L_p into L_r is continuous By Theorem I.5.8 it follows that if $f \in L(z)$ then $f \in L(U_{pr}z)$ and

$$U_{pr}(\int_E f dz) = \int_E f dU_{pr}z.$$

That is, if f is integrable with respect to z as a mapping into L_p for $1 \le p < \infty$, then f is integrable with respect to z as a mapping into L_r, $1 \le r \le p$, and the integrals coincide. Moreover, it is clear that the same sequence of step functions determines the integrals of f in L_p and in L_r.

By Theorem I.5.8 we have

$$E(\int_T f(t)z(dt))\overline{x} = \int_T f(t)E[z(dt)\overline{x}],$$

for every $x \in L_q$, since as is well known, this is the form of linear functionals on L_p (see Dunford and Schwartz [15]).

Thus

$$E(\int_T f(t)z(dt)) = \int_T f(t)E[z(dt)].$$

Also

$$E|\int_T f(t)z(dt)|^2 = E[\int_T f(t)z(dt)][\int_T \overline{f(\tau)z(d\tau)}]$$

$$= \int_T f(t)E[z(dt)\int_T \overline{f(\tau)z(d\tau)}].$$

In general

$$E[\int_T f(t)z(dt)]^k = \int_T f(t)E[z(dt)(\int_T \overline{f(\tau)z(d\tau)})^{k-1}],$$

since $[\int_T f(\tau)z(d\tau)]^{k-1}$ belongs to $L_{k/k-1}$ if $z:T \to L_k$. If we evaluate the scalar functions

$$t \to E[z(t)(\int_T \overline{f(\tau)z(d\tau)})^{k-1}],$$

then we can compute the k-th moment of the integral by performing a scalar integration. In practice this may be of some value.

4. Sample Path Integrals. Let $T = [0,1]$ and g be a continuous function on T. Let $z(t)$ be a Poisson process on T with parameter λ. As described in Section 2, almost every sample path $z(\cdot,\omega)$ is a monotone nondecreasing function, with $z(0,\omega) = 0$, and finitely many jumps of unit magnitude. We can therefore define the Riemann-Stieltjes integral

$$\int_0^1 g(t)z(dt,\omega)$$

for almost every ω. By Theorem II.2.3 it follows that

$$\int_0^1 g(t)z(dt,\omega) = \lim_{n\to\infty} \sum_{i=1}^{2^n} g(i-1/2^n)[z(i/2^n,\omega) - z(i-1/2^n,\omega)]$$

for almost every ω. Since the summands $g(i-1/2^n)[z(i/2^n) - z(i-1/2^n)]$ are random variables it follows that the sample

path integral $\int_0^1 g(t)z(dt)$ is also a random variable. Moreover, we can evaluate the integral directly, due to the simple behavior of the sample functions. Fix ω and let $n_o = z(1,\omega)$. Let $t_1,\ldots,t_{n_o}$ denote the n_o points of jump of the sample function $z(\cdot,\omega)$. For every n and i, the difference $z(i/2^n,\omega) - z(i-1/2^n,\omega)$ is either 1 or 0, depending on whether one of the t_i's belongs to $(i-1/2^n, i/2^n]$ or not. By the continuity of g we see that

$$\int_0^1 g(t)z(dt,\omega) = \sum_{i=1}^{n_o} g(t_i).$$

In general, denote by $t_i(\omega)$ the point in T of the i-th jump of the sample function $z(\cdot,\omega)$; that is, for $t \le t_i(\omega)$, $z(t,\omega) \le i-1$, while for $t > t_i(\omega)$, $z(t,\omega) \ge i$. Since $[\omega: t_i(\omega) < a] = [z(a) \ge i] \in F$ for each $a \in T$, t_i is a random variable for each i. Thus

$$\int_0^1 g(t)z(dt,\omega) = \sum_{i=1}^{z(1,\omega)} g(t_i(\omega)).$$

That is, the integral of g with respect to the Poisson process z is the sum of a random number $(z(1,\omega))$ of random variables of the form $g(t_i(\omega))$.

We can formally compute the mean of the integral as follows: we have

$$E\left[\sum_{i=1}^{2^n} g(i-1/2^n)[z(i/2^n)-z(i-1/2^n)]\right] = \sum_{i=1}^{2^n} g(i-1/2^n)\lambda(1/2^n).$$

Since $\lim \sum_{i=1}^{2^n} g(i-1/2^n)\lambda(1/2^n) = \lambda\int_0^1 g(t)dt$, and since the functions $\sum_{i=1}^{2^n} g(i-1/2^n)\chi_{[i-1/2^n,1/2^n)}$ converge uniformly to g on T, we would expect that

$$E\left[\int_0^1 g(t)z(dt)\right] = \lambda\int_0^1 g(t)dt.$$

This will follow once it is established that the stochastic Stieltjes integral exists. Proceeding to the formal computation of the variance of the integral, we compute

$$E\left|\sum_{i=1}^{2^n} g(i-1/2^n)[z(i/2^n) - z(i-1/2^n)]\right|^2 =$$

$$= \sum_{i=1}^{2^n}\sum_{j=1}^{2^n} g(i-1/2^n)\overline{g(j-1/2^n)}\,E[z(i/2^n)-z(i-1/2^n)] \times$$

$$\times [z(j/2^n)-z(j-1/2^n)]$$

$$= \sum_{i=1}^{2^n} |g(i-1/2^n)|^2\, E[z(i/2^n)-z(i-1/2^n)]^2 +$$

$$+ \sum_{i\neq j=1}^{2^n} g(i-1/2^n)\overline{g(j-1/2^n)}\,E[z(i/2^n)-z(i-1/2^n)] \times$$

$$\times E[z(j/2^n) - z(j-1/2^n)],$$

since z has independent increments. To evaluate the second moments, we suppose that x is a Poisson random variable with parameter α. Then

$$Ex^2 = \sum_{n=0}^{\infty} n^2 e^{-\alpha}\frac{\alpha^n}{n!} = \alpha e^{-\alpha}\sum_{n=1}^{\infty} n\frac{\alpha^{n-1}}{(n-1)!}$$

$$= \alpha e^{-\alpha}\left[\sum_{n=1}^{\infty}(n-1)\frac{\alpha^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty}\frac{\alpha^{n-1}}{(n-1)!}\right]$$

$$= \alpha e^{-\alpha}[\alpha e^{\alpha} + e^{\alpha}] = \alpha(\alpha + 1).$$

Since $z(i/2^n) - z(i-1/2^n)$ is a Poisson random variable with parameter $\lambda 2^{-n}$, we conclude

$$E[z(i/2^n) - z(i-1/2^n)]^2 = \lambda^2(1/2^n)^2 + \lambda(1/2^n).$$

Therefore,

$$E\left|\sum_{i=1}^{2^n} g(i-1/2^n)[z(i/2^n) - z(i-1/2^n)]\right|^2 =$$

$$= \sum_{i=1}^{2^n}\sum_{j=1}^{2^n} g(i-1/2^n)\overline{g(j-1/2^n)}\lambda^2(1/2^n)^2 + \sum_{i=1}^{2^n}|g(i-1/2^n)|^2\lambda(1/2^n).$$

The double sum converges to

$$\lambda^2\int_0^1\int_0^1 g(t)\overline{g(s)}\,dtds = \lambda^2\left(\int_0^1 g(t)dt\right)\left(\int_0^1 \overline{g(s)ds}\right)$$

$$= \lambda^2\left|\int_0^1 g(t)dt\right|^2.$$

The sum on the right converges to $\lambda\int_0^1|g(t)|^2dt$. We conclude that if the integral $\int_0^1 g(t)z(dt)$ exists as a norm integral in L_2, then

$$\sigma^2\left(\int_0^1 g(t)z(dt)\right) = \lambda^2\left[\left|\int_0^1 g(t)dt\right|^2\right] + \lambda\int_0^1|g(t)|^2dt -$$

$$- \left|\lambda\int_0^1 g(t)dt\right|^2$$

$$= \lambda\int_0^1|g(t)|^2dt.$$

Now since g is continuous we need only show that z is of weak bounded variation as a mapping into L_2; then the norm integral $\int_0^1 g(t)z(dt)$ belongs to L_2, hence to L_1 as well. By computations as above,

$$E\left|\sum_{i=1}^{n} z(t_i) - z(s_i)\right|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda^2(t_i - s_i)(t_j - s_j)$$

$$+ \sum_{i=1}^{n}\lambda(t_i - s_i)$$

$$\leq \lambda^2 + \lambda,$$

since $0 \le s_i \le t_i \le 1$ for each i. We conclude that z is of weak bounded variation by Remark II.2.2. Since the simple functions $\sum_{i=1}^{2^n} g(i-1/2^n)\chi_{[i-1/2^n, i/2^n)}$ converge uniformly to g on T, we conclude that

$$\int_0^1 g(t)z(dt) = \lim \sum_{i=1}^{2^n} g(i-1/2^n)[z(i/2^n) - z(i-1/2^n)]$$

in L_2 and hence in L_1. The expressions for the mean and variance now follow.

Motivated by this example, we can obviously define sample path integrals under the following conditions:

(i) $f:T \to \Phi$ is a continuous function and z(t) is a process with almost every sample function of bounded variation.

(ii) $f:T \to \Phi$ is a function of bounded variation and z(t) is a process with almost every sample function a continuous function.

If either (i) or (ii) hold then by the formula for integration by parts for Riemann-Stieltjes integrals (see Rudin [26]) we have

$$(*) \quad \int_0^1 f(t)z(dt,\omega) + \int_0^1 z(t,\omega)f(dt) = [z(\cdot,\omega)f(\cdot)]_0^1,$$

where both integrals are Riemann-Stieltjes integrals as in Definition II.2.4, and (*) holds for almost every ω.

In case (i), random variables of the form

$$\sum_{i=1}^{n} f(i-1/n)[z(i/n) - z(i-1/n)]$$

converge to $\int_0^1 f(t)z(dt)$ for almost every ω. In case (ii),

sums of the form

$$\sum_{i=1}^{n} z(i-1/n)[f(i/n) - f(i-1/n)]$$

converge to $\int_0^1 z(t)f(dt)$ for almost every ω. In both cases, then, the integral exists as a measurable function on (Ω,F,P); that is, the integral is a random variable.

In general, the existence and computation of the moments of a sample path integral depend on the properties of the functions $Ez(t)$, $Ez(t)\overline{z(s)}$, and so on, as we saw in the case of the integral with respect to a Poisson process.

A case of particular interest concerns random functions whose sample functions are nonnegative and nondecreasing functions. Since $s < t$ implies that $0 \le z(s) \le z(t)$ almost everywhere, we see that $0 \le Ez(s) \le Ez(t) \le Ez(1)$. If we suppose that $z(1)$ belongs to L_1, then it follows at once that z is of weak bounded variation, since the mapping $t \to Ez(t)$ is nondecreasing. If $f \in L(z)$, then we know from Theorem I.5.8 that $E\int_0^1 f(t)z(dt) = \int_0^1 f(t)Ez(dt)$. The Poisson integral discussed previously is a special case of this, since $Ez(t) = \lambda t$.

We now discuss an interesting connection between norm integrals and certain sample path integrals. Suppose that $z:T \to L_p$ is a random function of weak bounded variation, and suppose that z is also measurable with respect to the product measurable space $(T\times\Omega,\ \Sigma_1\times F)$. ($\Sigma_1\times F$ denotes the smallest σ-algebra containing all rectangles $A\times B$ with $A \in \Sigma_1$ and $B \in F$). Shachtman [27] asserts that the following integration by parts theorem holds, when $p = 1$.

4.1 Theorem. If $f:T \to \mathbb{C}$ is a function of bounded variation and $z:T \to L_1$ is product measurable and of weak bounded variation, then $f \in L(z)$, and

$$\int_A f(t)z(dt) = \int_A z(t)m_f(dt),$$

where the integral on the right is the sample path integral with respect to the negative of the Lebesgue-Stieltjes measure generated by f. A is taken to be an interval if m_z is not countably additive, and a Borel set if m_z is countably additive.

The proof of this theorem (Shachtman [27]) discusses only the case when $A = T$. That the theorem is false as stated is trivial, since it fails even if z is a nonrandom function, for example $z(t) = t$. For if f is constant then $m_f \equiv 0$. The question remains whether the formula

$$\int_0^1 f(t)z(dt) = \int_0^1 z(t)m_f(dt) + [zf]_0^1$$

can be true under these hypotheses. We establish the following result.

4.2 Theorem. Suppose $f:T \to \mathbb{C}$ is a continuous function of bounded variation, and $z:T \to L_p$ is product measurable and of weak bounded variation, where $1 \le p < \infty$. Then

$$\int_0^1 f(t)z(dt) + \int_0^1 z(t)m(dt) = [zf]_0^1,$$

where the integral in the center is the sample path integral of z with respect to the Borel-Stieltjes m on Σ_1 generated by f.

Proof. Since z is of weak bounded variation and f is continuous, $f \in L(z)$. z is a bounded function into L_p, so $\sup_T E|z(t)|^p < \infty$. Since the function $t \to E|z(t)|^p$ is measurable by the Fubini Theorem (see Dunford and Schwartz [15]), we have by this same theorem,

$$E\left(\int_0^1 |z(t)|^p v(m,dt)\right) = \int_0^1 E|z(t)|^p v(m,dt) < \infty.$$

Therefore the function $\omega \to \int_0^1 |z(t,\omega)|^p v(m,dt)$ is F-measurable and finite almost everywhere. Since the function $t \to |t|^p$ is convex, the Jessen inequality for integrals implies that for almost every ω, we have

$$\left|\frac{1}{v(m,T)} \int_0^1 |z(t,\omega)|v(m,dt)\right|^p \le \frac{1}{v(m,T)} \int_0^1 |z(t,\omega)|^p v(m,dt).$$

Therefore

$$E\left|\int_0^1 |z(t,\omega)|v(m,dt)\right|^p \le v(m,T)^{p-1} E\int_0^1 |z(t,\omega)|^p v(m,dt) < \infty.$$

We conclude that the mapping $\omega \to \int_0^1 z(t,\omega)m(dt)$ is a random variable in L_p.

Now since z is of weak bounded variation, each function $t \to Ez(t)\overline{x}$ is bounded, for $x \in L_q$. By an application of Fubini's Theorem we conclude that

$$\int_0^1 E[z(t)\overline{x}]m(dt) = E\left(\int_0^1 z(t)m(dt)\right)\overline{x},$$

where $\int_0^1 z(t)m(dt)$ denotes the sample path integral whose existence was proved above.

Since $f \in L(z)$, we know by Theorem I.5.8 that

$$E\left(\int_0^1 f(t)z(dt)\right)\overline{x} = \int_0^1 f(t)E[z(dt)\overline{x}],$$

for each $x \in L_q$. Using the integration by parts theorem (see Rudin [26]) we have

$$\int_0^1 f(t)E[z(dt)\overline{x}] = [f(\cdot)Ez(\cdot)\overline{x}]_0^1 - \int_0^1 E[z(t)\overline{x}]m(dt),$$

for each $x \in L_q$. Therefore

$$E\left(\int_0^1 f(t)z(dt)\right)\overline{x} = E[[fz]_0^1]\overline{x} - E\left(\int_0^1 z(t)m(dt)\right)\overline{x}$$

$$= E\left[[fz]_0^1 - \int_0^1 z(t)m(dt)\right]\overline{x}$$

for each $x \in L_q$. We conclude that

$$\int_0^1 f(t)z(dt) + \int_0^1 z(t)m(dt) = [fz]_0^1$$

as desired, where the second integral is the sample path integral of z in L_p. □

4. Processes with Orthogonal Increments. In Section II.4 we defined the concept of orthogonal increments for a function $z: T \to H$, where H is a Hilbert space. In particular, then, when $H = L_2(\Omega, F, P)$, the results of Section II.4 apply. Recall that the inner product in L_2 is defined by $\langle x,y \rangle = Ex\overline{y}$. Note that if x and y are independent random variables with $Ex = 0 = Ey$, then x and y are orthogonal since $Ex\overline{y} = ExE\overline{y} = 0$. It follows that the Wiener process is a process with orthogonal increments.

Wiener [29] used an approach to stochastic integration based on an integration by parts formula. Let $T = [0,1]$, and let m_1 denote Lebesgue measure in T; z denotes a Wiener process. Suppose that g is a function with a bounded derivative. Since almost every sample function of z is continuous, and hence bounded on T, Wiener defines

$$\int_0^1 g(t)z(dt,\omega) = [g(\cdot)z(\cdot,\omega)]_0^1 - \int_0^1 g'(t)z(t,\omega)dt.$$

The integral on the right exists for almost every ω since the integrand is bounded and measurable (g' is the limit of the

continuous functions $n[g(t+1/n) - g(t)]$). Using the orthogonality of the increments of the Wiener process, he then showed by somewhat lengthy computation that

$$E|\int_0^1 g(t)z(dt)|^2 = \int_0^1 |g(t)|^2 dt.$$

Using this identity and the fact that functions with a bounded derivative are dense in $L_2(m_1)$, Wiener then extends the definition of the integral to all of $L_2(m_1)$.

This process of defining stochastic integrals was simultaneously simplified and generalized by Doob [13]. He replaced the functions with bounded derivative by step functions, and defined the integral with respect to any process $z:T \to L_2$ with orthogonal increments. The relation

$$E|\int_0^1 g(t)z(dt)|^2 = \int_0^1 |g(t)|^2 m_1(dt),$$

where m_1 is a nonnegative measure generated by z as in Section II.4, and g is a step function, is central to the whole development. By the results of Section II.4 it follows that the Wiener-Doob integral in L_2 is contained in our general stochastic integration theory. Using the relation

$$(*) \quad E(\int_A f(t)z(dt))(\overline{\int_B g(t)z(dt)}) = \int_{A\cap B} f(t)\overline{g(t)}m_1(dt),$$

which is valid for $f,g \in L(m) = L_2^o(m_1)$ and $A,B \in \Sigma$, it follows that if we define a new processes $x:T \to L_2$ by

$$x(t) = \int_0^t f(\tau)z(d\tau),$$

where $f \in L(m)$, then x is a process with orthogonal increments. For if $s_1 < t_1 \le s_2 < t_2$, then we have

$$E(x(t_2) - x(s_2))(\overline{x(t_1)-x(s_1)}) = E(\int_{s_2}^{t_2} f(t)z(dt))(\overline{\int_{s_1}^{t_1} f(t)z(dt)})$$
$$= \int_{[s_1,t_1)\cap[s_2,t_2)} |f(t)|^2 m_1(dt) = 0.$$

More generally, if z is left continuous, then m_z and m_1 have countably additive extensions to Σ_1. Since the class of step functions is dense in $L(z)$ by Theorem I.8.3, relation (*) is still valid. It follows by this identity that the measure $\mu:\Sigma_1 \to L_2$ defined by

$$\mu(A) = \int_A f(t)z(dt),$$

$A \in \Sigma_1$, is a measure with orthogonal values.

Stochastic integrals with respect to processes with orthogonal increments have been used extensively in applications of probability theory to engineering. They are of particular value when the associated scalar measure m_1 can be determined, as is the case when z is a Wiener process. The measure m_1 allows us to compute the second moment of the stochastic integral of any f in $L(z)$, by simply computing the scalar integral $\int_0^1 |f|^2 dm_1$.

6. Martingale Integrals. In this section we shall introduce the concept of conditional expectation, and use it to define an important class of random functions, the martingales. Using a deep result of Burkholder [8], we show that every martingale $z:T \to L_p$ is of weak bounded variation, provided $1 < p < \infty$. Consequently every such random function determines a class of integrable functions containing all uniform limits of step functions. Since unbounded measurable functions may also be integrable with respect to a particular random function, our integral is somewhat more general in this respect than the martingale integral of Millar [24]. The Millar integral is not restricted to scalar-valued integrands, however.

Suppose x is a random variable in L_1, and $F_1 \subseteq F$ is a σ-algebra. The indefinite integral of x is a P-continuous, countably additive measure on F_1; by the Radon-Nikodým Theorem there is an integrable random variable z which is measurable with respect to F_1, unique up to a P-null set in F_1, and satisfies

$$\int_A z\,dP = \int_A x\,dP$$

for every $A \in F_1$. The random variable z is denoted by $E[x|F_1]$, and is called the conditional expectation of x with respect to F_1. The operator $E[\cdot|F_1]$ maps L_1 into L_1, and can be shown to be a continuous, positive, linear contraction operator.

Suppose that A is any set in F_1. Then we have, for every $B \in F_1$,

$$\begin{aligned} \int_B E[x\chi_A|F_1]\,dP &= \int_B x\chi_A\,dP \\ &= \int_{A\cap B} x\,dP \\ &= \int_B \chi_A E[x|F_1]\,dP. \end{aligned}$$

It follows that $E[x\chi_A|F_1] = \chi_A E[x|F_1]$ almost everywhere. By linearity and continuity of the operator $E[\cdot|F_1]$, we conclude that for any function y that is measurable with respect to F_1 and such that $E[xy|F_1]$ exists, we have $E[xy|F_1] = yE[x|F_1]$ almost everywhere. In particular, $E[y|F_1] = yE[1|F_1] = y$ almost everywhere, so $E[\cdot|F_1]$ is a projection operator.

Another useful property of the conditional expectation operator will now be established. Suppose that F_2 is another σ-algebra, and $F_2 \subseteq F_1 \subseteq F$. Then

$$\int_A E(E[x|F_2]|F_1)dP = \int_A E[x|F_2]dP$$

$$= \int_A x dP = \int_A E[x|F_1]dP$$

for every $A \in F_2$, so we conclude that

$$E[E[x|F_2]|F_1] = E[x|F_2] = E[E[x|F_1]|F_2],$$

up to some null set in F_2. Finally, since

$$Ex = \int_\Omega x dP = \int_\Omega E[x|F_1]dP,$$

we conclude that $E[E[x|F_1]] = Ex$.

Let $T = [0,1]$. We now define the martingale property.

6.1 Definition. Suppose $1 \le p \le \infty$ and $z:T \to L_p$ is a random function. Suppose that $(F(t):t \in T)$ is a family of σ-algebras contained in F such that $F(s) \subseteq F(t)$ if $s \le t$. Then $z(t)$ is a martingale with respect to $(F(t):t \in T)$ if $z(t)$ is measurable with respect to $F(t)$ for each $t \in T$, and if $s < t$, then

(*) $E[z(t)|F(s)] = z(s)$ almost everywhere.

We say for short that $(z(t),F(t))$ is a martingale. If $(z(t),F(t))$ is a martingale and $t_1 < t_2 < \ldots < t_n$ is any finite set of points in T, then $(z(t_i),F(t_i):1 \le i \le n)$ is a martingale in the sense that (*) holds for s and t restricted to the set $\{t_1,\ldots,t_n\}$. In this section we shall consider only real-valued random functions and scalar functions.

An important result concerning conditional expectations is the conditional Jessen inequality. If $g:R \to R$ is a convex

function, if $x \in L_1$, and if $g(x) \in L_1$, then for every σ-algebra F_1 contained in F, we have $g(E[x|F_1]) \le E[g(x)|F_1]$ almost everywhere. For a proof of this result see Chung [9]..

Suppose that $z:T \to L_p$ is a martingale with respect to $(F(t):t \in T)$. Since $t \to |t|^p$ is a convex function for every $p \ge 0$, it follows that for $s \le t$,

$$|z(s)|^p = |E[z(t)|F(s)]|^p \le E[|z(t)|^p|F(s)],$$

almost everywhere. Hence $E|z(s)|^p \le E|z(t)|^p$, using the property $E[E[x|F]] = Ex$ of conditional expectations. We conclude that the family of p-th absolute moments of a martingale in L_p is a nondecreasing family. Since $E|z(t)|^p \le E|z(1)|^p$ for every t, we see that z is a bounded function from T to L_p.

To show that any martingale z is actually of weak bounded variation when $1 < p < \infty$, we need to introduce the concept of a martingale transform.

6.2 <u>Definition</u>. Suppose that (z_n,F_n) is a martingale in L_p, and (a_n) is a sequence of real numbers. We say that a sequence (x_n) of random variables is the martingale transform of (z_n) by (a_n) if for every n,

$$x_n = \sum_{i=1}^{n} a_i[z_i - z_{i-1}],$$

where we set $z_0 = 0$.

If (x_n) is the transform of (z_n), then it follows that (x_n,F_n) is a martingale. For x_n is measurable with respect

to F_n, since each z_i, $1 \le i \le n$, is measurable with respect to F_n. Moreover,

$$E[x_{n+k}|F_n] = \sum_{i=1}^{n+k} a_i E[z_i - z_{i-1}|F_n]$$

$$= \sum_{i=n+1}^{n+k} a_i E[z_i - z_{i-1}|F_n] + \sum_{i=1}^{n} a_i E[z_i - z_{i-1}|F_n]$$

$$= \sum_{i=n+1}^{n+k} a_i [z_n - z_n] + \sum_{i=1}^{n} a_i [z_i - z_{i-1}]$$

$= x_n$ almost everywhere.

The following theorem is due to Burkholder [8].

6.3 <u>Theorem</u>. (Burkholder) For each p, $1 < p < \infty$, there is a constant K_p such that if (x_n) is the transform of (z_n, F_n) by a sequence (a_n), with $|a_n| \le 1$ for all n, then

$$E|x_n|^p \le K_p E|z_n|^p, \quad n = 1,2,\ldots .$$

The proof of this result, which depends on showing, for each p, the equicontinuity of the sequence of bounded linear operators:

$$T_n(\cdot) = a_1 E[\cdot|F_1] + \sum_{i=2}^{n} a_i\{E[\cdot|F_i] - E[\cdot|F_{i-1}]\},$$

will not be given.

We can now state our final result.

6.4 <u>Theorem</u>. Suppose $1 < p < \infty$ and $z:T \to L_p$ is a martingale with respect to $(F(t):t \in T)$. Then

(i) z is of weak bounded variation.

(ii) If $f \in L(z)$ then $E(\int_0^1 f(t)z(dt)) = 0$.

(iii) If $f \in L(z)$ and we define

$$x(t) = \int_0^t f(\tau)z(d\tau),$$

for $t \in T$, then $(x(t), F(t):t \in T)$ is a martingale.

Proof. (i) Suppose that $t_1 < t_2 < \ldots < t_{2n}$ is any finite family in T. Then $(z(t_i), F(t_i):1 \le i \le n)$ is a martingale. Let $a_{2i} = 1$ and $a_{2i-1} = 0$ for $1 \le i \le n$. Then

$$E\left|\sum_{i=1}^{n} z(t_{2i}) - z(t_{2i-1})\right|^p = E\left|\sum_{i=1}^{2n} a_i[z(t_i) - z(t_{i-1})]\right|^p$$

$$\le K_p E|z(t_{2n})|^p \le K_p E|z(1)|^p < \infty,$$

by Theorem 6.3. By Remark II.2.2, it follows that z is of weak bounded variation.

(ii) Since $z(t) = E[z(1)|F(t)]$ for every $t \in T$, we have $Ez(t) = E[E[z(1)|F(t)]] = Ez(1)$. Hence $Em_z[a,b) = E[z(b) - z(a)] = 0$ for every interval in T. It follows that for every step function f on T, $E\int_0^1 f(t)z(dt) = 0$. Since the class of step functions is dense in L(z), (ii) holds.

(iii) Since $f \in L(z)$, there is a sequence of step functions f_k such that the indefinite integrals of the f_k's converge uniformly for $A \in \Sigma$ to the indefinite integral of f. Define

$$x_k(t) = \int_0^t f_k(\tau)z(d\tau)$$

for each k and every $t \in T$. Then $\lim x_k(t) = x(t)$ for every t, the limit taking place in L_p. It suffices to show that each x_k is a martingale with respect to $(F(t), t \in T)$. For if this is the case, then by continuity of the conditional expec-

tation operator, we have

$$E[x(t)|F(s)] = \lim_k E[x_k(t)|F(s)]$$

$$= \lim_k x_k(s) = x(s).$$

Suppose then that f is a step function of the form

$$f = \sum_{i=1}^{n-1} a_i \chi_{[t_{i-1},t_i)} + a_n \chi_{[t_{n-1},t_n]},$$

where $0 = t_0 < t_1 < \ldots < t_n = 1$. If $t \in T$, then

$$x(t) = \int_0^t f(\tau)z(d\tau)$$

$$= \sum_{i=1}^{k-1} a_i[z(t_i) - z(t_{i-1})] + a_k[z(t) - z(t_{k-1})],$$

where $t \in [t_{k-1},t_k)$. Now suppose that $s < t$. Then

$$E[x(t)|F(s)] = \sum_{i=1}^{k} a_i E[z(t_i) - z(t_{i-1})|F(s)]$$

$$= \sum_{i=j+1}^{k} a_i[z(s) - z(s)] + a_j E[z(t_j)-z(t_{j-1})|F(s)]$$

$$+ \sum_{i=1}^{j-1} a_i[z(t_i) - z(t_{i-1})]$$

$$= a_j[E[z(t_j) - z(s)|F(s)] + E[z(s)-z(t_{j-1})|F(s)]]$$

$$+ \sum_{i=1}^{j-1} a_i[z(t_i) - z(t_{i-1})]$$

$$= a_j[z(s) - z(t_{j-1})] + \sum_{i=1}^{j-1} a_i[z(t_i)-z(t_{i-1})]$$

$$= x(s), \text{ where } s \in [t_{j-1},t_j).$$

We conclude that $(x(t),F(t))$ is a martingale. □

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